Computer Science 341
Discrete Mathematics

Final Exam

Guidelines: No collaboration is permitted on the final. You may refer to your own notes, textbooks (including those other than the two course texts), and all materials posted on the course web site. The use of any other electronic materials and searching for information online is not allowed.

Any fact/lemma/theorem in the course materials and course texts may be used in your solutions without proof. If you use material from a textbook other than the course texts, you should include an appropriate citation with your solution; additionally the use of any fact/lemma/theorem from these additional sources must be accompanied with a proof.

There are 5 problems and each problem is worth 20 points. You have 24 hours to complete the exam. Since students will be taking the exam at different times, please do not discuss the questions on the final after you are done.

The solution for each problem must begin on a new page. Be concise and clear. Excessively verbose solutions may be penalized.

All the best!

Problem 1

Consider strings of A's, B's and C's consisting of n A's and a total of $2n$ B's and C's (i.e the number of B's plus the number of C's is $2n$). Further, there are no consecutive occurences of the same character. How many such strings are there?

Problem 2

Consider a complete binary tree. We color the vertices of the tree with two colors, red and blue. A valid coloring is one where every internal red node has both its children colored blue (does not apply if the node is a leaf). Let C_n be the number of valid colorings of a complete binary tree of height n. Thus, $C_0 = 2$, $C_1 = 5$, and so on

1. Show that $C_n = C_{n-1}^2 + C_{n-2}^4$.
2. Show that there exist constants α and β such that $C_n \leq \alpha \cdot (\beta)^{2^n}$.

Problem 3

Consider a finite set of points with a distance function $d(x, y)$ that specifies a distance for every pair of points x, y. The distance function satisfies the triangle inequality, i.e., $d(x, z) \leq d(x, y) + d(y, z)$. Suppose that $d(x, y) = 0$ or 1 for all x, y. Prove that you can divide the set of points into disjoint clusters such that $d(x, y) = 0$ for points in the same cluster and $d(x, y) = 1$ for points in different clusters.

Problem 4

Consider a random bipartite graph $G(V_1, V_2, E)$ where $|V_1| = |V_2| = n$, and each edge in $V_1 \times V_2$ is present with probability c/n. Here c is a constant.

0. Suppose all edges in $V_1 \times V_2$ are present. How many perfect matchings does G contain? (Note: we are asking about how many distinct perfect matchings there are. These matchings need not be disjoint.)

1. Find the expected number of perfect matchings in G as a function of c.

2. Find a value c_0 such that, as $n \to \infty$, the expected number of perfect matchings is less than 1 for any constant $c < c_0$ and greater than 1 for any constant $c > c_0$.

<u>Problem 5</u>

Consider a graph G on n vertices that has no cycle of length $\leq 2k+1$. Let m be the number of edges in the graph. The goal of this problem is to prove that $m \leq n^{1+1/k} + n$.

0. What is the average degree in G?

1. Prove that there exists a subgraph H of G with minimum degree m/n. *(Hint: Think about vertices that have degree less than m/n.)*

2. Let v be a vertex in H. Consider the subgraph of H induced by vertices at distance at most k from v. Prove that this subgraph is a tree.

3. Prove that $m/n \leq n^{1/k} + 1$. *(Hint: Give bounds on the number of vertices in the subgraph constructed in part 2.)*