Reductions

Linear time reductions

Polynomial time reductions

NP-completeness

Reduction

Problem X reduces to problem Y if given a subroutine for Y, can solve X.

- Cost of solving X = cost of solving Y + cost of reduction.
- May call subroutine for Y more than once.

Ex: X = baseball elimination, Y = max flow.

Consequences:

- Establish relative difficulty between two problems. (classify problems)
- Given algorithm for Y, can also solve X. (design algorithms)
- If X is hard, then so is Y. (establish intractability)

Linear Time Reductions

Problem X linear reduces to problem Y if X can be solved with:

- Linear number of standard computational steps.
- One call to subroutine for Y.
- Notation: $X \leq_L Y$.

more generally, if given a O(f(N)) time subroutine for Y, can solve X in O(f(N)) time

Examples we've already seen in the course.

- Removing duplicates reduces to sorting.
- Voronoi diagram reduces to Delaunay triangulation.
- Arbitrage reduces to negative cycle detection.
- Bipartite matching reduces to max flow.
- Brewer's problem reduces to linear programming.

Other most common type of reduction.

- X polynomial reduces to Y.
- Stay tuned for NP-completeness.

Linear Time Reduction: Examples

PRIME: Given (the decimal representation of) an integer x, is x prime?

COMPOSITE: Given an integer x, does x have a nontrivial factor?

FACTOR: Given two integers x and y, does x have a nontrivial factor less than y?

other than 1 and x

Claim. COMPOSITE $\leq_L$ PRIME.

composite (x)

```
if (prime(x)) return false;
else          return true;
```

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Linear Time Reduction: Examples

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COMPOSITE: Given an integer x, does x have a nontrivial factor?
FACTOR: Given two integers x and y, does x have a nontrivial factor less than y?

other than 1 and x

Claim. COMPOSITE $\leq_L$ FACTOR.

- Is 62773913 composite?
- Does 62773913 have a nontrivial factor less than 62773913?
- Yes, 62773912 = 7919 × 7927.

composite (x)

```
if (factor(x, x)) return true;
else              return false;
```


Problem Equivalence

Tool for classifying problems.

- Equivalence: If $X \leq_L Y$ and $Y \leq_L X$ then we write $X \equiv_L Y$.
  - given any algorithm for X, can solve Y in same running time, and vice versa
- Transitivity: if $X \leq_L Y$ and $Y \leq_L Z$ then $X \leq_L Z$.

Equivalence: PRIME $\leq_L$ COMPOSITE and COMPOSITE $\leq_L$ PRIME.

Transitivity: PRIME $\leq_L$ COMPOSITE $\leq_L$ FACTOR.

Primality Testing and Factoring

We established: PRIME $\leq_L$ FACTOR.

Natural question: Does FACTOR $\leq_L$ PRIME ?

- Consensus opinion = no.

State-of-the-art.

- PRIME is in P.
- FACTOR not believed to be in P.

RSA cryptosystem.

- Based on dichotomy between two problems.
- To use RSA, must generate large primes efficiently.
- Can break RSA with efficient factoring algorithm.

Reduction Gone Wrong

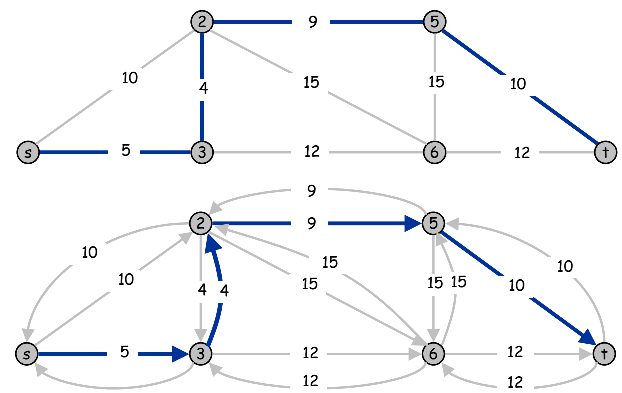
Caveat.
- System designer specs the interfaces for project.
- One programmer might implement `isComposite` using `isPrime`.
- Another programmer might implement `isPrime` using `isComposite`.
- Be careful to avoid infinite reduction loops in practice.

```java
public static boolean isComposite(int x) {
   if (isPrime(x)) return false;
   else            return true;
}
```

```java
public static boolean isPrime(int x) {
   if (isComposite(x)) return false;
   else                return true;
}
```

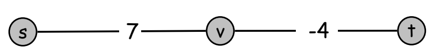

Undirected Shortest Path Reduces to Directed Shortest Path

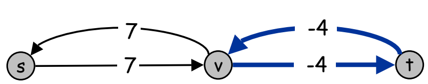
Undirected shortest path (with nonnegative weights) linearly reduces to directed shortest path.
- Replace each directed arc by two undirected arcs.
- Shortest directed path will use each edge at most once.



Shortest Path with Negative Costs

Caveat: Reduction invalid in networks with negative cost arcs, even if no negative cycles.





Remark: can still solve shortest path problem in undirected graphs if no negative cycles, but need more sophisticated techniques.
- Reduce to weighted non-bipartite matching. (!)

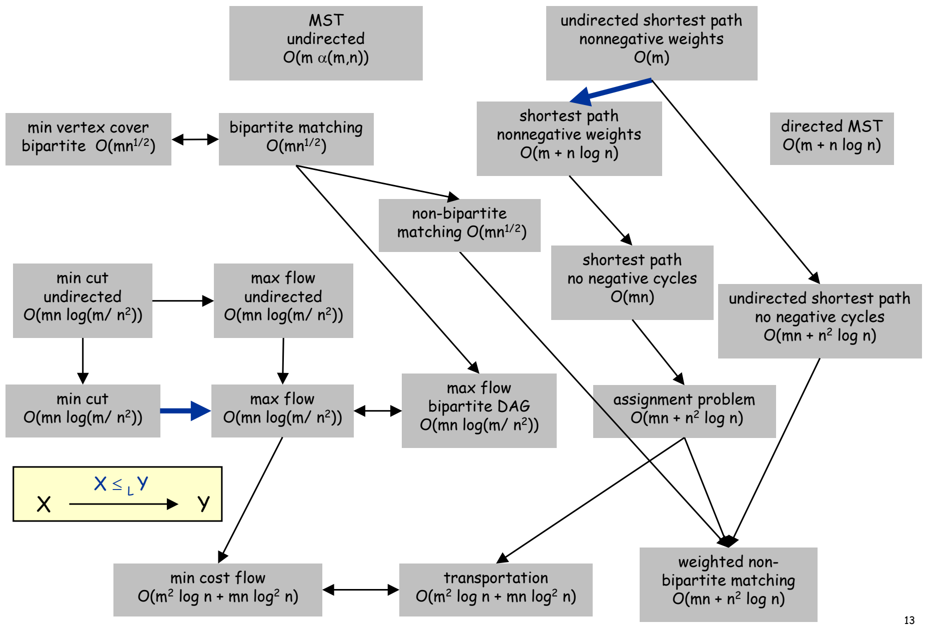
Reduction: Min Cut Reduces to Max Flow

Max-flow min-cut theorem says value of max flow = capacity of min cut.

Min cut linear reduces to max flow.
- Given a max flow, find all vertices reachable from source in residual graph to get min cut.

Does max flow linear reduce to min cut?
- Apparently no easy way to determine max flow from min cut.
- But no better way known to compute a min cut than via max flow.

Network Flow Running Times and Linear Time Reductions



Integer Arithmetic

Integer multiplication: given two N-digit integer s and t, compute s × t.

Integer division: given two integers s and t of at most N digits each, compute the quotient $q = \lfloor s / t \rfloor$ and remainder r = s mod t.

Operation	Grade School	Best Known Lower Bound
Addition	O(N)	Ω(N)
Multiplication	O(N²)	Ω(N)
Division	O(N²)	Ω(N)

Fundamental questions.
- Is multiplication easier than division?
- Is addition easier than multiplication?
- Is division easier than multiplication?

Integer Arithmetic

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Operation	Grade School	Best Known Upper Bound
Addition	O(N)	O(N)
Multiplication	O(N²)	O(N log N log log N)
Division	O(N²)	O(N log N log log N)

Theorem. Integer multiplication and integer division have the same asymptotic complexity.
- Multiplication linear reduces to division.
- Division linear reduces to multiplication.

Sorting and Convex Hull

Sorting. Given N distinct integers, rearrange in increasing order.

Convex hull. Given N points in the plane, find their convex hull in counter-clockwise order.

Lower bounds.
- Recall, under comparison-based model of computation, sorting N items requires Ω(N log N) comparisons.
- We show sorting linearly reduces to convex hull.
- Hence, finding convex hull of N points requires Ω(N log N) "comparisons" where comparison means `ccw`.

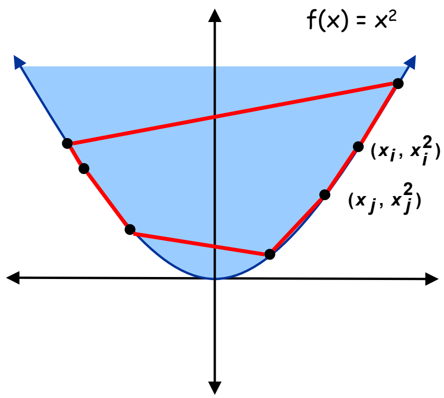
Sorting Reduces to Convex Hull

Sorting instance (integers):

$x_1, x_2, \ldots, x_N$

Convex hull instance:

$(x_1, x_1^2),\ (x_2, x_2^2), \ldots, (x_N, x_N^2)$



Key observation.

- Region $\{x : x^2 \geq x\}$ is convex $\Rightarrow$ all points are on hull.
- Starting at point with most negative x, counter-clockwise order of convex hull yields items in sorted order.

3-SUM Reduces to 3-COLLINEAR

3-SUM: Given N distinct integers $x_1, x_2, \ldots x_N$, are there 3 distinct integers x_i, x_j, x_k such that $x_i + x_j + x_k = 0$?

3-COLLINEAR: Given N distinct points $(x_1, y_1), (x_2, y_2), \ldots (x_N, y_N)$, are there 3 points that all lie on the same line?

pattern recognition assignment

Conjecture: Any algorithm for 3-SUM requires $\Omega(N^2)$ time.

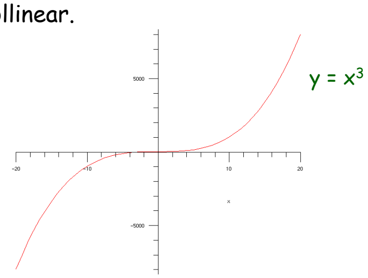
Claim. 3-SUM $\leq_L$ 3-COLLINEAR.

Corollary. Unless you can solve 3-SUM is sub-quadratic time, any algorithm for 3-COLLINEAR requires $\Omega(N^2)$ time.

Reduction. To determine if there is a solution to 3-SUM instance $x_1, x_2, \ldots x_N$, determine if there is a solution to 3-COLLINEAR instance with $(x_1, x_1^3), (x_2, x_2^3), \ldots, (x_N, x_N^3)$.

3-SUM Reduces to 3-COLLINEAR

Claim. If a, b, and c are distinct then a + b + c = 0 if and only if (a, a^3), (b, b^3), (c, c^3) are collinear.



Proof. Necessary and sufficient conditions for two line segments to be equal.

$$
\begin{aligned}
\frac{a^3-b^3}{a-b} = \frac{b^3-c^3}{b-c} &\Leftrightarrow \frac{(a-b)(a^2+ab+b^2)}{a-b} = \frac{(b-c)(b^2+bc+c^2)}{b-c} \\
&\Leftrightarrow c^2+bc-a^2-ab=0 \\
&\Leftrightarrow (c-a)(c+a+b)=0 \\
&\Leftrightarrow c=a \quad \text{or} \quad a+b+c=0
\end{aligned}
$$

Polynomial-Time Reduction

X polynomial reduces to Y if X can be solved using:

- Polynomial number of standard computational steps.
- Polynomial number of calls to subroutine for Y.
- Notation: $X \leq_P Y$.

Alternate viewpoint. Can solve X in polynomial time given special piece of hardware that solves instances of Y in a single step.

no different from polynomial in this context

Ex: Baseball elimination reduces to max flow.

- Solve N max flow problems on a graph with N^2 vertices.

Remark 1: If $X \leq_L Y$ then $X \leq_P Y$.

Remark 2: If X can be solved in polynomial time, then $X \leq_P Y$ for any Y.

Polynomial-Time Reduction

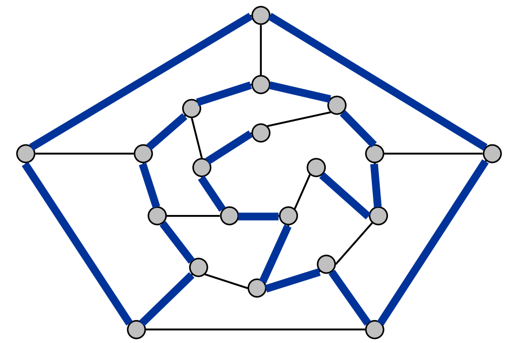
Goal: classify and separate problems according to relative difficulty.
- Those that can be solved in polynomial time.
- Those that (probably) require exponential time.

Establish tractability. If $X \leq_P Y$ and Y can be solved in polynomial-time, then X can be solved in polynomial time.

Establish intractability. If $X \leq_P Y$ and X cannot be solved in polynomial-time, then Y cannot be solved in polynomial time.

Hamilton Path

HAMILTON-PATH. Given an undirected graph, is there a path that visits every vertex exactly once?



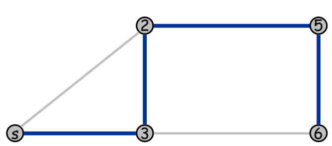
EULER-PATH. Given an undirected graph, is there a path that visits every edge exactly once?

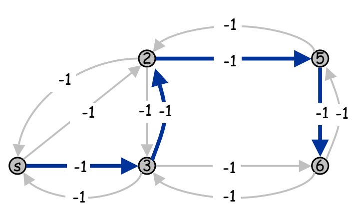
Hamilton Path Reduces to Shortest Path

HAMILTON-PATH. Given an undirected graph, is there a path that visits every vertex exactly once?

SHORTEST-PATH. Given an directed network and two vertices s and t, find the shortest simple path from s to t.

Claim. HAMILTON-PATH $\leq_P$ SHORTEST-PATH.
- For each undirected edge, make two directed edges of weight -1.
- For all pairs of vertices v and w, find shortest path from v to w.
- If shortest path has length -(V-1) then this is a Hamilton path.





Hamilton Path Reduces to Shortest Path

Claim. HAMILTON-PATH $\leq_P$ SHORTEST-PATH.

Conjecture. No polynomial algorithm exists for HAMILTON-PATH.

Corollary. Polynomial algorithm for SHORTEST-PATH is unlikely.
- This explains why we needed the "no negative cycles" assumption for shortest path algorithms.

Shortest Path	Algorithm	Running Time
Nonnegative weights	Dijkstra	E log V
No negative cycles	Bellman-Ford	E V
Arbitrary weights	Brute force	2^V

Subset Sum Reduces To Integer Programming

SUBSET-SUM. Given N integers $a_1, a_2, \ldots a_N$, and another integer b, is there a subset of integers that sums to exactly b?

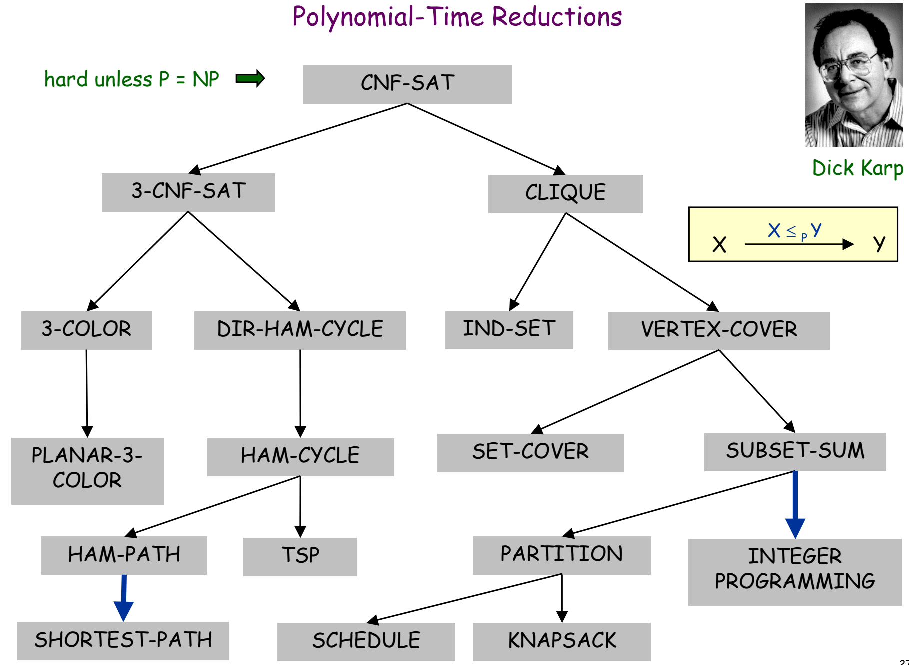
Integer programming. Given integers b_i, a_{ij} find 0/1 variables x_i that satisfy a linear system of equations.

$$\begin{aligned} \sum_{j=1}^{N} a_{ij} x_j &= b_i & 1 \le i \le M \\ x_j &\in \{0, 1\} & 1 \le j \le N \end{aligned}$$

SUBSET-SUM polynomial reduces to IP. Solve integer program below and select subset of indices with $x_i = 1$.

$$\begin{aligned} \sum_{j=1}^{N} a_j x_j &= b \\ x_j &\in \{0, 1\} & 1 \le j \le N \end{aligned}$$

Polynomial-Time Reductions



NP-Completeness

P. Set of all decision problems solvable in polynomial time on a deterministic Turing machine.

NP. Set of all decision problems solvable in polynomial time on a nondeterministic Turing machine.

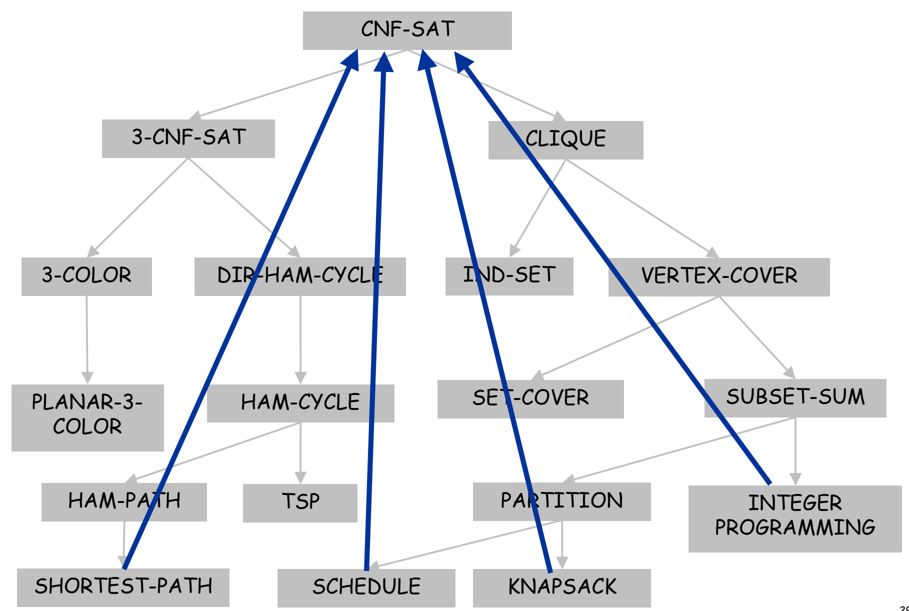
NP-complete. Decision problem X is NP-complete if every problem in NP polynomial reduces to X.

Cook's theorem. CNF-SAT is NP-complete.

Corollary. If $P \ne NP$, then no polynomial algorithm for CNF-SAT.

Practical consequence. If $P \ne NP$, then can't hope to design polynomial algorithm for any problem on the previous slide.

Polynomial-Time Reductions



Summary

Reductions are important in theory to:

- Classify problems according to their computational requirements.
- Establish intractability.
- Establish tractability.

Reductions are important in practice to:

- Design algorithms.
- Design reusable software modules.
  - sorting, priority queue, symbol table, graph, shortest path, max flow, regular expressions, linear programming
- Determine difficulty of your problem and choose the right tool.
  - use exact algorithm for tractable problems
  - use heuristics for NP-hard problems (e.g., bin packing)