Binary Search Trees

Binary search trees

Randomized BSTs

Reference: Chapter 12, Algorithms in Java, 3rd Edition, Robert Sedgewick.

Guaranteeing Performance

Symbol table: key-value pair abstraction.

- Insert a value with specified key.
- Search for value given key.
- Delete value with given key.

Challenge 1: guarantee symbol table performance.

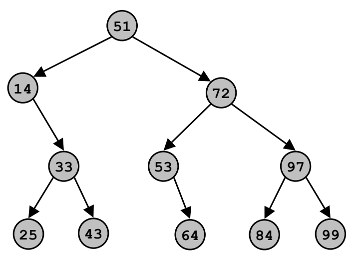
- Make average case independent of input distribution.
- Extend average case guarantee to worst-case.
- Remove assumption on having a good hash function.
- Remove expensive (but infrequent) re-doubling operations.

Challenge 2: expand interface when keys are ordered.

- Find the i^{th} largest key.
- Range searching.

Binary Search Tree

Binary search tree: binary tree in symmetric order.

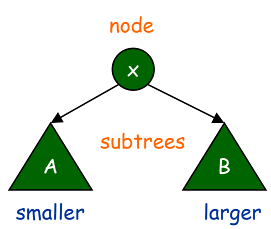


Binary tree is either:

- Empty.
- A key-value pair and two binary trees.

Symmetric order:

- Keys in nodes.
- No smaller than left subtree.
- No larger than right subtree.



Binary Search Tree in Java

A BST is a reference to a node.

A Node is comprised of four fields:

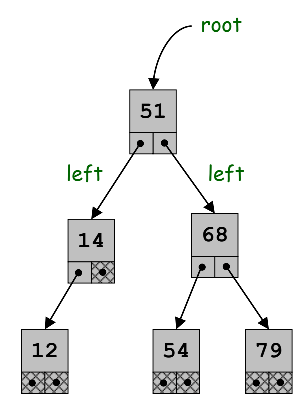
- A key and a value.
- A reference to the left and right subtree.

smaller larger

```
private static class Node {
   Comparable key;
   Object value;
   Node left;
   Node right;
}
```

key can be any Comparable object

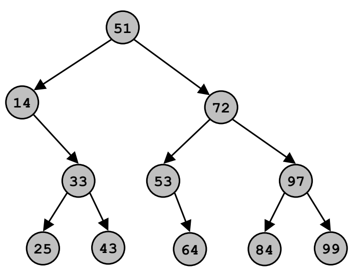
helper class

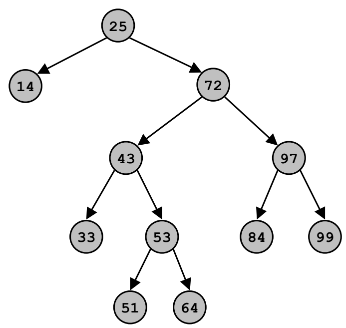


Tree Shape

Tree shape.

- Many BSTs correspond to same input data.
- Have different tree shapes.
- Performance depends on shape.





BST Skeleton

```
public class SymbolTable {
   private Node root;

   private static class Node {
      Comparable key;
      Object value;
      Node left, right;

      Node(Comparable key, Object value) {
         this.key = key;
         this.value = value;
      }
   }

   private static boolean   less(Comparable k1, Comparable k2) { }
   private static boolean equals(Comparable k1, Comparable k2) { }

   public void put(Comparable key, Object value) { }
   public Object get(Comparable key) { }
}
```

helper inner class

helper functions

ST interface methods

BST Search

Search for specified `key` and return corresponding `value` or `null`.

- Code follows from BST definition.
- Use helper function to search for key in subtree rooted at h.

```
public Object get(Comparable key) {
   return search(root, key);
}

private Object search(Node h, Comparable key) {
   if (h == null)            return null;          ⇐ not found
   if (equals(key, h.key))   return h.value;       ⇐ found
   if (less  (key, h.key))   return search(h.left,  key);
   else                      return search(h.right, key);
}
```

go left or right

BST Insert

Insert `key`-`value` pair.

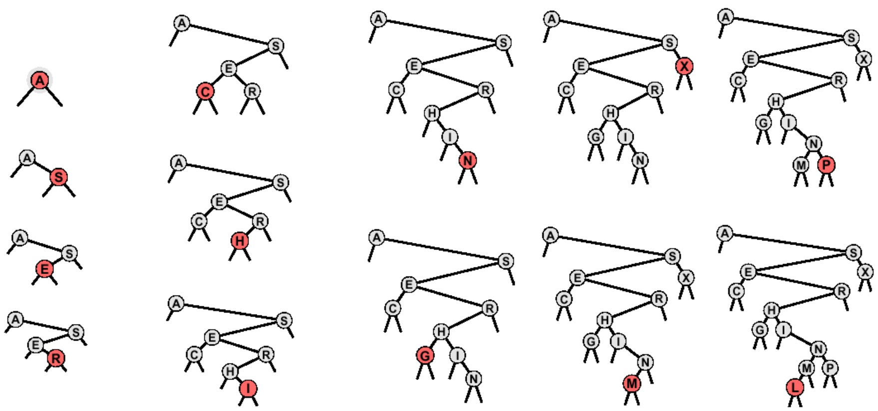
- Code follows from BST definition.
- Search, then insert.
- Simple (but tricky) recursive code.
- Duplicates allowed.

```
public void put(Comparable key, Object value) {
   root = insert(root, key, value);
}

private Node insert(Node h, Comparable key, Object value) {
   if (h == null) return new Node(key, value);
   if (less(key, h.key))  h.left  = insert(h.left,  key, value);
   else                   h.right = insert(h.right, key, value);
   return h;
}
```

BST Construction

Insert the following keys into BST: **A S E R C H I N G X M P L**



BST Analysis

Cost of search and insert BST.

- Proportional to depth of node.
- 1-1 correspondence between BST and quicksort partitioning.
- Height of node corresponds to number of function calls on stack when node is partitioned.

Theorem. If keys are inserted in random order, then height of tree is $\Theta(\log N)$, except with exponentially small probability. Thus, search and insert take $O(\log N)$ time.

Problem. Worst-case search and insert are proportional to N.

- If nodes in order, tree degenerates to linked list.

Symbol Table: Implementations Cost Summary

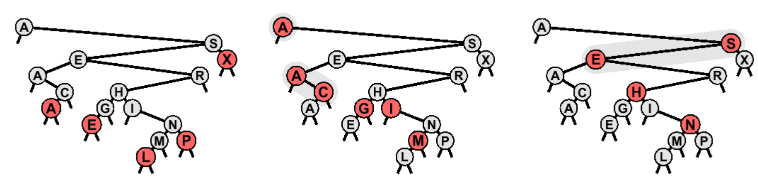
	Worst Case			Average Case		
Implementation	Search	Insert	Delete	Search	Insert	Delete
Sorted array	log N	N	N	log N	N / 2	N / 2
Unsorted list	N	1	1	N / 2	1	1
Hashing	N	1	N	1*	1*	1*
BST	N	N	N	log N	log N	???

BST: log N insert and search if keys arrive in random order.
Ahead: Can we make all ops log N if keys arrive in arbitrary order?

Symbol Table: Delete

To delete a node:

- Case 1 (zero children): just remove it.
- Case 2 (one child): pass the child up.
- Case 3 (two children): find the next largest node using right-left* or left-right*, swap with next largest, remove as in Case 1 or 2.



Case 1 Case 2 Case 3

Problem: strategy clumsy, not symmetric.
Serious problem: trees not random (!!)

Symbol Table: Implementations Cost Summary

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Implementation	Search	Insert	Delete	Search	Insert	Delete
Sorted array	log N	N	N	log N	N / 2	N / 2
Unsorted list	N	1	1	N / 2	1	1
Hashing	N	1	N	1*	1*	1*
BST	N	N	N	log N	log N	sqrt(N) †

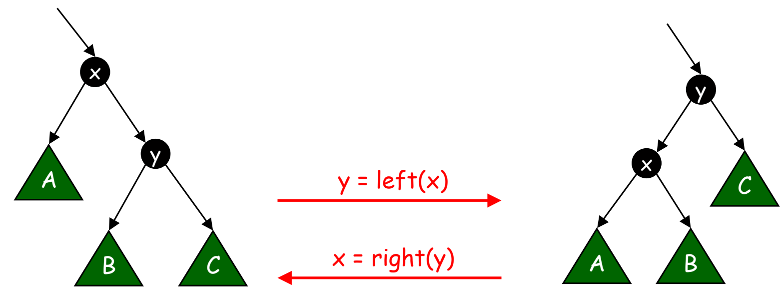
* assumes our hash function can generate random values for all keys
† if delete allowed, insert/search become sqrt(N) too

Ahead: Can we achieve log N delete?
Ahead: Can we achieve log N worst-case?

Right Rotate, Left Rotate

Fundamental operation to rearrange nodes in a tree.

- Maintains BST order.
- Local transformations, change just 3 pointers.



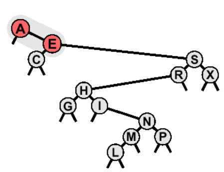
Right Rotate, Left Rotate

Fundamental operation to rearrange nodes in a tree.

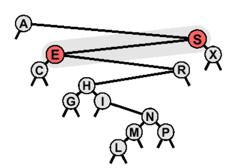
- Easier done than said.

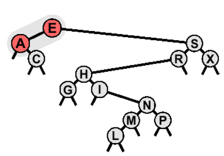
Left rotate

Right rotate

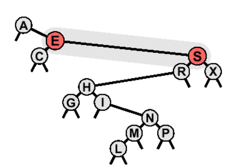


```
private Node rotL(Node h) {
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   return x;
}
```





```
private Node rotR(Node h) {
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   return x;
}
```



Recursive BST Root Insertion

Root insertion: insert a node and make it the new root.

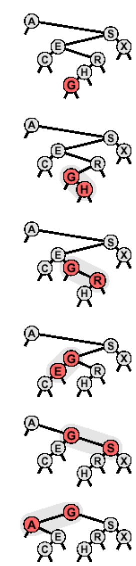
- Insert the node using standard BST.
- Use rotations to bring it up to the root.

Why bother?

- Faster if searches are for recently inserted keys.
- Basis for advanced algorithms.

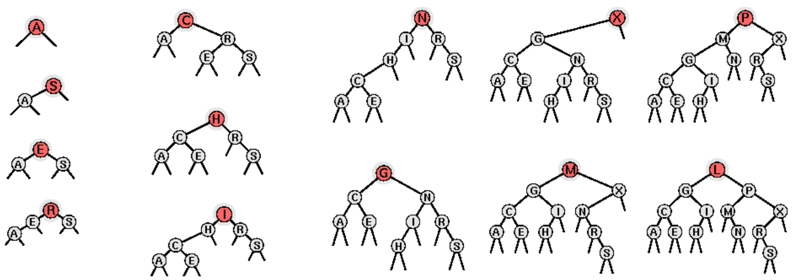
```
Node insertT(Node h, Comparable key, Object value) {
   if (h == null) return new Node(key, value);
   if (less(key, h.key)) {
      h.left = insertT(h.left, key, value);
      h = rotR(h);
   }
   else {
      h.right = insertT(h.right, key, value);
      h = rotL(h);
   }
   return h;
}
```

insert G



BST Construction: Root Insertion

A S E R C H I N G X M P L



Randomized BST

Observation. If keys are inserted in random order then BST is balanced with high probability.

Idea. When inserting a new node, make it the root (via root insertion) with probability 1/(N+1) and do it recursively.

```
private Node insert(Node h, Comparable key, Object value) {
   if (h == null) return new Node(key, value);
   if (Math.random()*(h.N+1) < 1) return insertT(h, key, value);
   if (less(key, h.key))  h.left  = insert(h.left,  key, value);
   else                   h.right = insert(h.right, key, value);
   h.N++;
   return h;
}
```

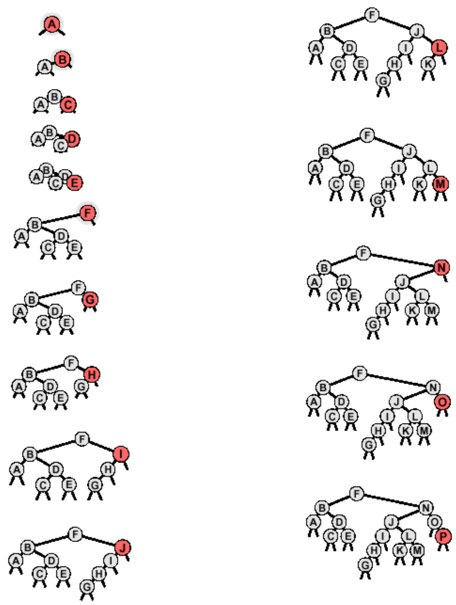
Fact. Tree shape distribution is identical to tree shape of inserting keys in random order.

- No assumptions made on the input distribution!

Randomized BST Example

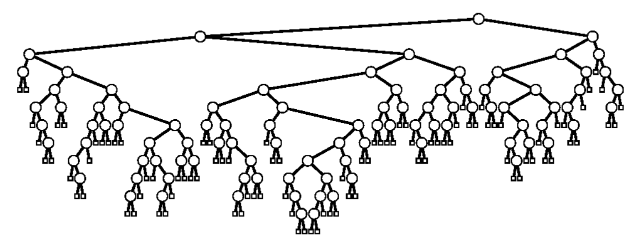
Insert keys in order.

- Tree shape still random.



Randomized BST

Always "looks like" random binary tree.



- Implementation: maintain subtree size in each node.
- Supports all symbol table ops.
- log N average case.
- Exponentially small chance of bad balance.

Randomized BST: Delete

Join. Merge two disjoint symbol tables A (of size M) and B (of size N), assuming all keys in A are less than all keys in B.

- Use A as root with probability M / (M + N), and recursively join right subtree of A with B
- Use B as root with probability N / (M + N), and recursively join left subtree of B with A

Delete. Given a key k, delete and return a node with key k.

- Delete the node.
- Join two broken subtrees as above.

Theorem. Tree still random after delete.

Symbol Table: Implementations Cost Summary

	Worst Case			Average Case		
Implementation	Search	Insert	Delete	Search	Insert	Delete
Sorted array	log N	N	N	log N	N / 2	N / 2
Unsorted list	N	1	1	N / 2	1	1
Hashing	N	1	N	1*	1*	1*
BST	N	N	N	log N	log N	sqrt(N) †
Randomized BST	log N ‡	log N ‡	log N ‡	log N	log N	log N

* assumes our hash function can generate random values for all keys
† if delete allowed, insert/search become sqrt(N)
‡ assumes system can generate random numbers

Randomized BST: guaranteed log N performance!
Next time: Can we achieve deterministic guarantee?

BST: Other Operations

Sort. Traverse tree in ascending order.

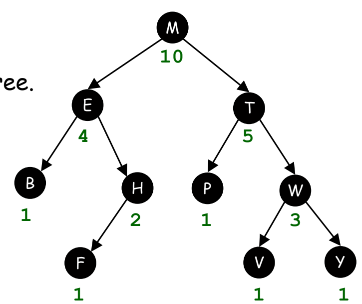
- Inorder traversal.
- Same comparisons as quicksort, but pay space for extra links.

Range search. Find all items whose keys are between k_1 and k_2.

Find k^{th} largest. Generalized PQ that finds k^{th} smallest.

- Special case: find min, find max.
- Add subtree size to each node.
- Takes time proportional to height of tree.

```
private class Node {
   Comparable key;
   Object value;
   Node left, right;
   int N;
}        subtree size
```



Randomized BST: Other Operations

Ceiling. Given key k, return smallest element that is at least as big as k.

Best-fit bin packing heuristic. Insert the item in the bin with the least remaining space among those that can store the item.

Theorem. Best-fit decreasing is guaranteed use no more than 11B/9 + 1 bins, where B is the best possible.

- within 22% of best possible.
- original proof of this result was over 70 pages of analysis!

Symbol Table: Implementations Cost Summary

	Worst Case Asymptotic Cost						
Implementation	Search	Insert	Delete	Find k^{th}	Sort	Join	Ceil
Sorted array	log N	N	N	log N	N	N	log N
Unsorted list	N	1	1	N	N log N	N	N
Hashing	1*	1*	1*	N	N log N	N	N
BST	N	N	N	N	N	N	N
Randomized BST	log N ‡	log N ‡	log N ‡	log N ‡	log N ‡	log N ‡	log N ‡

* assumes our hash function can generate random values for all keys
‡ assumes system can generate random numbers

makes no assumption on input distribution

Randomized BST: O(log N) average case for ALL ops!

Next time: Can we achieve deterministic guarantee?