Symmetry Detection

Motivation

- Compression
- Reconstruction
- Classification
- Analysis
- Alignment
- Matching
- Etc.
Definition

A collection of:
- points/lines/curves/triangles/surfaces/volumes,
has **reflective symmetry** w.r.t. some plane $p$ if
the reflection $\text{Ref}_p$ through $p$ fixes the
collection.

Definition

A collection of:
- points/lines/curves/triangles/surfaces/volumes,
has **rotational symmetry** of order $k$ w.r.t.
some axis $p$ if the rotation $\text{Rot}_p^k$ by an angle
of $360^\circ/k$ about $p$ fixes the collection.
Outline

- Geometry based approaches
- Geometry/descriptor based approaches
- Descriptor based approaches

Discrete Symmetry

String matching

\[ S = ABACABAC \]
Discrete (Rotational) Symmetry

Search for non-trivial repeating patterns in concatenation ($S \cdot S$):

$$S \cdot S = \begin{array}{cccccccccccc}
\end{array}$$

Discrete (Reflective) Symmetry

Search for non-trivial repeating patterns in concatenation ($S^t \cdot S$):

$$S^t \cdot S = \begin{array}{cccccccccccc}
\end{array}$$
Discrete (Reflective) Symmetry

Search for non-trivial repeating patterns in concatenation ($S^t_\_ S\cdot S$):

\[
S\cdot S = \begin{bmatrix}
\end{bmatrix}
\]

Outline

- Geometry based approaches
- Geometry/descriptor based approaches
- Descriptor based approaches
Points on a Circle

Sort by angle and compute angle between adjacent points

\[ S = \alpha \alpha \beta \chi \beta \]

Generate string from ordered list of angles

IEEE, 1985. Atallah
Information Processing Letters, 1986. Highnam

Points on a Circle

Find rotational and reflective symmetries of string:

Rotational Symmetry

\[ S \cdot S = \alpha \alpha \beta \chi \beta \]

Reflective Symmetry

\[ S \cdot S' = \beta \chi \beta \alpha \alpha \beta \chi \alpha \alpha \beta \chi \alpha \alpha \beta \chi \]

\[ S' \]
Points in 2D

Discussion

Pro:
- Evaluates all symmetries

Cons:
- Only perfect symmetry
- Does not generalize to 3D

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<tr>
<th>Feature</th>
<th>3D</th>
<th>Continuous Measure of Symmetry</th>
<th>Identifies All Symmetries</th>
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Symmetry Distance

Measure of symmetry as distance to nearest symmetric model:

Initial model | Nearest 3-fold symmetric | Symmetry Distance


Symmetry Distance

Nearest symmetric model can be obtained by folding:

Symmetry Distance

Needs establishment of correspondences

Discussion

Pro:
- A continuous measure of symmetry
- Generalizes to 3D

Cons:
- Does not identify potential symmetries
- Depends on establishing of correspondences

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Outline

- Geometry based approaches
- Geometry/descriptor based approaches
- Descriptor based approaches

Approach

Leverage shape descriptor to obtain a structured shape representation:
- Correspondences become implicit
PCA

PCA of a 3D model gives an ellipsoid:


Symmetry of the Ellipsoid

Planes of reflective symmetry of an ellipsoid are perpendicular to principal axes

Symmetry of the Ellipsoid

Axes of rotational symmetry of an ellipsoid:
• Are principal axes
• Only occur if the other two axes have the same length (when order $g = 2$)

Computing Symmetry with PCA

1. Compute the principal axes of a model
2. Identify candidate axes/planes of symmetry
3. Evaluate quality of candidates by comparing the shape descriptor (SD) of initial models with the SDs of the rotations/reflections.

Discussion

Pros:
• A continuous measure of symmetry
• Generalizes to 3D
• Addresses the correspondence issue

Con:
• Evaluates each symmetry type independently

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Outline

• Geometry based approaches
• Geometry/descriptor based approaches
• Descriptor based approaches
**Approach**

Leverage shape descriptor to obtain a structured shape representation:

- Possibility for efficient exhaustive symmetry detection

![Diagram of 3D Shape to 2D Shape and Descriptor]

**Observation**

Sub-string based symmetry detection is efficient but is binary.

![Diagram of Sub-string matching with S]
Circular Function Descriptors

Replace discrete matching of circular string with correlation of circular function

Correlation is $O(b \log(b))$, with $|SD| = O(b)$

$Sym(SD, \alpha) = \int_0^{2\pi} SD(t - \alpha)SD(t)dt$

PRL, 1995. Sun
RTI, 1999. Sun et al.

2D Function Descriptors

Generalize to 2D functions by looking at circular restrictions

Correlation is $O(b^2 \log(b))$, with $|SD| = O(b^2)$

$Sym(SD, \gamma) = \sum_{i=1}^{3} \frac{Sym(SD, \gamma)_{1}}{r_{i}} + \frac{Sym(SD, \gamma)_{2}}{r_{2}} + \frac{Sym(SD, \gamma)_{3}}{r_{3}} + \ldots$

Spherical Function Descriptors

Use spherical harmonics and Wigner-D transform to perform rotational correlation.

Shape Descriptor (SD)

$$\text{Sym}(SD, R) = \int_{\text{Sphere}} SD(R(t))R(t)dt$$

Correlation is $O(b^4)$, with $|SD| = O(b^2)$

3D Function Descriptors

Generalize to 3D functions by looking at spherical restrictions

$$\text{Sym}(SD, R) = r_1 \text{Sym}^{(1)}(R) + r_2 \text{Sym}^{(2)}(R) + r_3 \text{Sym}^{(3)}(R) + \ldots$$

Correlation is $O(b^4)$, with $|SD| = O(b^3)$

Summary

- Use shape descriptors for efficiency and simplicity
- Generalize fast sub-string matching to FFT
- Extend FFT to FST

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