Solving Ordinary Differential Equations with Matlab and Maple

October 25, 2000

1 A sample problem

We consider the initial value problem:

\[
\frac{d\mathbf{x}(t)}{dt} = f(t, \mathbf{x}(t)) \tag{1}
\]

\[
\mathbf{x}_0 = \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{2}
\]

where

\[
\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \quad f(t, \mathbf{x}(t)) = \begin{pmatrix} -\alpha x(t) + y(t) \\ -x(t) - \alpha y(t) \\ -z^2(t)/(t + e) \end{pmatrix} \tag{3}
\]

The exact solution is

\[
\mathbf{x}(t) = \begin{pmatrix} e^{-\alpha t} \sin(t) \\ e^{-\alpha t} \cos(t) \\ \ln^{-1}(t + e) \end{pmatrix} \tag{4}
\]

2 Solving with Matlab

The main ODE solvers for non-stiff problems are ode23 and ode45. Here we exhibit by means of a simple example the use of the latter, which is a higher order method (ode23 can be used instead).

First we need a mfile that implements \( f(t, \mathbf{x}(t)) \). This should be a Matlab function that receives as input a parameter \( T \) and a \( 3 \times 1 \) vector \( \mathbf{X} \). The output is a \( 3 \times 1 \) vector that corresponds to \( f \) evaluated at \((T, \mathbf{X})\).
function F=stable(T,X)
% F=stable(T,X)

F = zeros(2,1);  % F must be a column vector
a = 0.1;
F(1) = X(2) - a*X(1);
F(2) = - (X(1) + a*X(2)) - (X(3)^2)/(T + exp(1));

Alternatively one may write

function F=stable(T,X)
% F=stable(T,X)

a = 0.1;
F = [ X(2) - a*X(1)
      - (X(1) + a*X(2)) - (X(3)^2)/(T + exp(1) ) ];

For consistency the name of the file matches the name of the function, ie
stable.m. Then we may call the above function in a Matlab environment, as
for example:

>> F=stable(2,[1 1 1])

Next we make the call to the ODE solver and plot the results. For this
purpose we use the following script file script.m:

X0 = [0; 1; 1];  % initial values
Tspan = [0 50];  % domain of the approximation
[T,X] = ode45('stable',Tspan,X0);  % solve using Matlab’s default precision
% T = Nx1 vector with the discrete time steps
% X = Nx3 matrix with the corresponding approximations
subplot(2,2,1), plot(T,X(:,1)), xlabel('t'), ylabel('x(t)'), axis tight;
subplot(2,2,2), plot(T,X(:,2)), xlabel('t'), ylabel('y(t)'), axis tight;
subplot(2,2,3), plot(T,X(:,3)), xlabel('t'), ylabel('z(t)'), axis tight;
subplot(2,2,4), plot3(X(:,1),X(:,2),X(:,3));
subplot(2,2,4), xlabel('x'), ylabel('y'), zlabel('z'), axis tight;
title('stable system');

We execute the script by typing:

>> script
>> print -deps stable.eps

The last command saves the figure as an eps file, that can be included in a
\LaTeXX file\footnote{of course there are other options - see help print}:

2
The diagrams illustrate the behavior of a stable system over time. The plots show the functions $x(t)$, $y(t)$, and $z(t)$ against time $t$. The system appears to converge to a stable state, indicating that it is a stable system.
Above a small example of instability:

\[
f(t, \mathbf{x}(t)) = \begin{pmatrix} -\alpha x(t) + y(t) \\ -x(t) - \alpha y(t) \\ 1/(t + e) \end{pmatrix}
\]  

This was just an overview. For the necessary details see the online manual's of Matlab for ode23, ode45, plot, plot3, print.
3 Solving with Maple

The first step is to assign to some variables the expressions that describe the differential equations. This is done in a interactive Maple environment as follows (Maple’s responses are also shown):

\[
\text{> ode1} := \text{diff(x(t),t)} = -a x(t) + y(t);
\]
\[
\text{ode1} := \frac{d}{dt} x(t) = -a x(t) + y(t)
\]

\[
\text{> ode2} := \text{diff(y(t),t)} = -x(t) - a y(t);
\]
\[
\text{ode2} := \frac{d}{dt} y(t) = -x(t) - a y(t)
\]

\[
\text{> ode3} := \text{diff(z(t),t)} = -(z(t))^{2}/(t+\exp(1));
\]
\[
\text{ode3} := \frac{d}{dt} z(t) = -(t+\exp(1))^{-2}
\]

We may assign the whole system to another variable:

\[
\text{> ODE} := \{\text{ode1,ode2,ode3}\};
\]
\[
\text{ODE} := \{\frac{d}{dt} x(t) = -a x(t) + y(t), \frac{d}{dt} y(t) = -x(t) - a y(t), \frac{d}{dt} z(t) = -(t+\exp(1))^{-2}\}
\]

or together with the initial conditions:

\[
\text{> InitVal} := \{\text{ode1,ode2,ode3,x(0)=0,y(0)=1,z(0)=1}\};
\]
\[
\text{InitVal} := \{\frac{d}{dt} x(t) = -a x(t) + y(t), \frac{d}{dt} y(t) = -x(t) - a y(t), \frac{d}{dt} z(t) = -(t+\exp(1))^{-2}, y(0) = 1, z(0) = 1\}
\]

Now everything is set and we may use Maple’s dsolve routine both for analytic and numerical solutions.

We seek the general solution of our system with no initial conditions specified:

\[
\text{> dsolve(ODE,\{x(t),y(t),z(t)\});}
\]
\[
\{x(t) = -C1 \exp(-t a) \sin(t) - C1 \exp(-t a) \cos(t),
\]

\[
y(t) = -C1 \exp(-t a) \cos(t) + C1 \exp(-t a) \sin(t),
\]

\[
z(t) = \exp(t+\exp(1)) (C1 \exp(-t a) \sin(t) + C1 \exp(-t a) \cos(t)),
\]

\[
C1 \text{ arbitrary constant.}
\]
\[
y(t) = -c_1 \exp(-t \ a) \sin(t) + -c_2 \exp(-t \ a) \cos(t), \quad \{z(t) = \text{--------------------------}\} \\
\ln(t + \exp(1)) + c_3
\]

Let's try a numerical solution

```maple
dsolve(ODE, \{x(t), y(t), z(t)\}, \text{type=numeric}); 
Error, (in dsolve/numeric/error_conditions) initial conditions missing
```

We forgot something! Another attempt

```maple
dx := dsolve(InitVal, \{x(t), y(t), z(t)\}, \text{type=numeric});
x := proc(rkf45_x) ...
end
```

Maple's responses can be weird sometimes! But now things are not so complicated: Maple has returned a list of equations describing the value of \( t \) and the values of \( x(t), y(t) \) and \( z(t) \) at that points:

```maple
> X(0);
[t = 0, x(t) = 0, y(t) = 1., z(t) = 1.]
> X(1);
Error, (in dsolve/numeric/rkf45) cannot evaluate boolean
```

In the second example we got a negative response because we didn't specify a value for the parameter \( a \). Correction:

```maple
> a := 0.1;
a := .1
```

```maple
> X(1);
[t = 1, x(t) = .7613944391615206, y(t) = .48885743682876, z(t) = .7614628579625739]
```

```maple
> X(50);
[t = 50, x(t) = -.00176785974209687, y(t) = .00650193461122264, z(t) = .2522092077102136]
```

Here is how we convert the numeric solution to functions, one for each coordinate:

```maple
> x := t -> rhs(op(2, X(t))); y := t -> rhs(op(3, X(t))); z := t -> rhs(op(4, X(t)))
```

Now we can plot each individual component:

```maple
> plot(x, 0..50);
```

For more information check the online manuals of Maple (usage: ?< topic >) for dsolve, diff, int, plot, eval, evalf.
References
