Spline
Curves & Surfaces

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3D Object Representations

- Raw data
  - Voxels
  - Point cloud
  - Range image
  - Polygons

- Surfaces
  - Mesh
  - Subdivision
  - Parametric
  - Implicit

- Solids
  - Octree
  - BSP tree
  - CSG
  - Sweep

- High-level structures
  - Scene graph
  - Skeleton
  - Application specific
Curved Surfaces

• Motivation
  ◦ Exact boundary representation for some objects
  ◦ More concise representation than polygonal mesh

Curved Surfaces

• What makes a good surface representation?
  ◦ Accurate
  ◦ Concise
  ◦ Intuitive specification
  ◦ Local support
  ◦ Affine invariant
  ◦ Arbitrary topology
  ◦ Guaranteed continuity
  ◦ Natural parameterization
  ◦ Efficient display
  ◦ Efficient intersections
Curve Representations

- Function
  - $y = f(x)$

- Implicit
  - $f(x, y) = 0$

- Parametric
  - $x = f(u)$
  - $y = f(u)$

- Subdivision
  - $(x, y, z)$ defined by limit of recursive process

Curved Surface Representations

- Function
  - $z = f(x,y)$

- Implicit
  - $f(x, y, z) = 0$

- Parametric
  - $x = f(u, v)$
  - $y = f(u, v)$
  - $z = f(u, v)$

- Subdivision
  - $(x, y, z)$ defined by limit of recursive process
Function Surface Representation

• Boundary defined by explicit function:
  ◦ \( z = f(x, y) \)

Implicit Surfaces

• Boundary defined by implicit function:
  ◦ \( f(x, y, z) = 0 \)

• Example: linear (plane)
  ◦ \( ax + by + cz + d = 0 \)
Implicit Surfaces

- Example: quadric
  \[ f(x,y,z) = ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fxz + 2gx + 2hy + 2jz + k \]

- Common quadric surfaces:
  - Sphere
  - Ellipsoid
  - Torus
  - Paraboloid
  - Hyperboloid

[Diagram of Implicit Surfaces]

H&B Figure 10.10

Parametric Surfaces

- Boundary defined by parametric function:
  \[
  \begin{align*}
  x &= f(u, v) \\
  y &= f(u, v) \\
  z &= f(u, v)
  \end{align*}
  \]

- Example (sphere):
  \[
  \begin{align*}
  x &= \cos(\theta)\cos(\phi) \\
  y &= \sin(\theta)\cos(\phi) \\
  z &= \sin(\phi)
  \end{align*}
  \]
**Subdivision Surfaces**

- Coarse mesh & subdivision rule
  - Define smooth surface as limit of sequence of refinements

**Outline**

- **Today:**
  - Parametric curves and surfaces

- **Wednesday:**
  - Subdivision surfaces

- **Next week:**
  - Implicit surfaces
### Parametric curves

A parametric curve in the plane is expressed as:

\[
\begin{align*}
  x &= x(u) \\
  y &= y(u)
\end{align*}
\]

**Example**: a circle with radius \( r \) centered at origin:

\[
\begin{align*}
  x &= r \cos u \\
  y &= r \sin u
\end{align*}
\]

### Parametric polynomial curves

- A parametric polynomial curve is described:

\[
\begin{align*}
  x(u) &= \sum_{i=0}^{n} a_i u^i \\
  y(u) &= \sum_{i=0}^{n} b_i u^i
\end{align*}
\]

- Advantages of polynomial curves
  - Easy to compute
  - Infinitely differentiable
### Piecewise Param Polynomial Curves

- **Idea:**
  - Use different polynomial functions on different parts of the curve

- **Advantage:**
  - Flexibility
  - Control

- **Issue:**
  - Smoothness at “joints”? *(continuity)*

### Continuity

- Continuity $C^k$ indicates adjacent curves have the same $k$th derivative at their joints
**C^0 Continuity**

- Adjacent curves share …
  - Same endpoints: \( Q_i(1) = Q_{i+1}(0) \)

**C^1 Continuity**

- Adjacent curves share …
  - Same endpoints: \( Q_i(1) = Q_{i+1}(0) \)
  - Same derivatives: \( Q_i'(1) = Q_{i+1}'(0) \)
**C² Continuity**

- Adjacent curves share …
  - Same endpoints: \( Q_i(1) = Q_{i+1}(0) \)
  - Same derivatives: \( Q_i'(1) = Q_{i+1}'(0) \)
  - Same second derivatives: \( Q_i''(1) = Q_{i+1}''(0) \)

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**Spline constructions**

- C² interpolating splines
- Hermite
- Bezier
- Catmull-Rom
- B-splines

Blending functions:

\[
Q(u) = \sum_{i=0}^{k} V_i b_i(u)
\]
C^2 Interpolating splines

- Blending functions are chosen so that …
  - Control points are interpolated
  - Adjacent curves meet with C^2 continuity

C^2 Interpolating splines

- Properties:
  - Interpolate control points
  - C^2 continuity
  - No local control
Spline constructions

- $C^2$ interpolating splines
- Hermite
- Bezier
- Catmull-Rom
  - B-splines

Uniform Cubic B-Splines

- Choose blending functions so that …
  - Cubic polynomials
  - $C^2$ continuity
  - Local control
  - Points not necessarily interpolated
Uniform Cubic B-Splines

• Derivation:
  ○ Three continuity conditions for each joint $J_i$ …
    » Position of two curves are equal at $J_i$
    » Derivatives of two curves are equal at $J_i$
    » Second derivatives of two curves are equal at $J_i$
  ○ Also, local control implies …
    » Each joint is affected by small set of (4) points

Uniform Cubic B-Splines

• Fifteen continuity constraints:

<table>
<thead>
<tr>
<th>Condition</th>
<th>Index 0</th>
<th>Index 1</th>
<th>Index 2</th>
<th>Index 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 = b_{i0}(0)$</td>
<td>$0 = b_{i0}'(0)$</td>
<td>$0 = b_{i0}''(0)$</td>
<td>$b_{i0}(0) + b_{i1}(0) + b_{i2}(0) + b_{i3}(0) = 1$</td>
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<tr>
<td>$b_{i0}(1) = b_{i1}(0)$</td>
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<td>$b_{i3}(1) = 0$</td>
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<td>$b_{i3}''(1) = 0$</td>
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</tr>
</tbody>
</table>
Uniform Cubic B-Splines

- Solving the system of equations:
  \[ b_{-3}(u) = -\frac{1}{6}u^3 + \frac{1}{2}u^2 - \frac{1}{2}u + \frac{1}{6} \]
  \[ b_{-2}(u) = \frac{1}{2}u^3 - u^2 + \frac{2}{3} \]
  \[ b_{-1}(u) = -\frac{1}{2}u^3 + \frac{1}{2}u^2 + \frac{1}{2}u + \frac{1}{6} \]
  \[ b_0(u) = \frac{1}{6}u^3 \]

- Matrix form for uniform cubic B-spline:
  \[
  Q(u) = \begin{bmatrix}
  u^3 & u^2 & u & 1
  \end{bmatrix}
  \begin{bmatrix}
  -\frac{1}{2} & 1/2 & -\frac{1}{2} & 1/6 \\
  1/6 & -\frac{1}{2} & 1/2 & 0 \\
  1/2 & -1 & 1/2 & 0 \\
  -1/2 & 0 & 1/2 & 0 \\
  1/6 & 2/3 & 1/6 & 0
  \end{bmatrix}
  \begin{bmatrix}
  V_{i-3} \\
  V_{i-2} \\
  V_{i-1} \\
  V_i
  \end{bmatrix}
  \]
Uniform Cubic B-Splines

- Properties:
  - $C^2$ continuity
  - Local control
  - Approximating (points not interpolated)
  - Convex hull property

Parametric Patches

- Each patch is defined by blending control points
Parametric Patches

- Point \( Q(u,v) \) on the patch is the tensor product of parametric curves defined by the control points.

\[ Q(u,v) = UM^{T}V^{T} \]

\[ U = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \quad V = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix} \]

Where \( M \) is a matrix describing the blending functions for a parametric cubic curve (e.g., Bezier, B-spline, etc.)
B-Spline Patches

\[ Q(u, v) = U M_{\text{B-Spline}} \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} M_{\text{B-Spline}}^T V \]

\[ M_{\text{B-Spline}} = \begin{bmatrix} -1/6 & 1/2 & -1/2 & 1/6 \\ 1/2 & -1 & 1/2 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 1/6 & 2/3 & 1/6 & 0 \end{bmatrix} \]

Watt Figure 6.28

Parametric Surfaces

- Advantages:
  - Easy to enumerate points on surface
  - Possible to describe complex shapes

- Disadvantages:
  - Control mesh must be quadrilaterals
  - Continuity constraints difficult to maintain
  - Hard to find intersections