Computer Science 341
Discrete Mathematics

Homework 7
Due in class on Wed, Nov 13, 2002

Collaboration Policy: You may collaborate in groups of at most 3 students. These groups must be disjoint and discussion across groups is not allowed. Collaboration is limited to discussion of ideas only, and you should write up the solutions entirely on your own and list your collaborators.

Problem 1
Suppose that G is a graph with $2n$ nodes and no triangles (cycles of length 3). G is a proper graph, i.e. it has no self-loops or multiple edges between the same pair of nodes. Prove that G has at most n^2 edges.

Problem 2
Let G be an undirected graph where each vertex has degree greater than or equal to k. Show that G contains a cycle of length greater than or equal to $k + 1$.

Problem 3
An Eulerian graph G is called *randomly Eulerian* from a vertex v if every walk in G starting from v that does not repeat an edge can be extended into an Eulerian tour.
(a) Prove that an Eulerian graph $G(V, E)$ is randomly Eulerian from a vertex v if and only if the graph $G - v$ contains no cycle. ($G - v$ denotes the graph obtained from v by removing vertex v and all edges incident on v).
(b) Show that if a connected graph G is randomly Eulerian from more than 2 vertices in G, then it is randomly Eulerian from all vertices in G.

Problem 4
1024 computers are connected to each other as follows: each one of them is assigned a unique 10 bit sequence and two computers are connected by a direct link if their corresponding bit sequences differ in exactly one position. If a machine fails, all links incident on it are unavailable for communication. We would like the network to be fault tolerant, i.e. remain connected even if there are a small number of machine failures. Prove that the minimum number of machine failures required to disconnect the network is 10. (*Hint*: Show that every two machines have 10 disjoint paths connecting them).

Problem 5
Let $T = (V, E)$ be a tree and v some vertex of T. Let $\tau(v) = \max(|V(T_1)|, |V(T_2)|, \ldots, |V(T_k)|)$, where $T_1, \ldots, T_k$ are all the components of the graph $T - v$. The centroid of the tree T is the set of all vertices $v \in V$ with the minimum value of $\tau(v)$.
(a) Prove that the centroid of any tree is either a single vertex or two vertices connected by an edge.
(b) Prove that if v is a vertex in the centroid then $\tau(v) \leq \frac{2}{3}|V(T)|$.