Computer Science 341
Discrete Mathematics

Homework 6
Due in class on Wed, Nov 6, 2002

Collaboration Policy: No collaboration is permitted for this homework.

Problem 1

An *automorphism* of a graph $G = (V, E)$ is any isomorphism of G and G, i.e. any bijection $f : V \to V$ such that $\{u, v\} \in E$ if and only if $\{f(u), f(v)\} \in E$. A graph is called *asymmetric* if its only automorphism is the identity mapping (each vertex is mapped to itself). Show that a graph G with n vertices is asymmetric if and only if $n!$ distinct graphs on the set $V(G)$ are isomorphic to G.

Problem 2

Show that if a function $d : V \times V \to N$ satisfies conditions 1–5 in page 108 of Matousek, then a graph $G = (V, E)$ exists such that $d_G(v, v') = d(v, v')$ for any pair of elements of V.

Problem 3

a. Find a connected graph of n vertices for which each of the powers $A_G^1, A_G^2, \ldots$ of the adjancency matrix contains some zero elements.

b. Let G be a graph on n vertices, $A = A_G$ its adjacency matrix, and I_n the $n \times n$ identity matrix (with 1s on the diagonal and 0s elsewhere). Prove that G is connected if and only if the matrix $(I_n + A)^{n-1}$ has no 0s.

c. Where are the 0s in the matrix $(I_n + A)^{n-1}$ if the graph G is not connected?

Problem 4

a. Construct an example of a sequence of length n in which each term is from the set $\{1, 2, \ldots, n-1\}$ and which has an even number of odd terms, and yet the sequence is not a graph score. (You must prove that the sequence is not a graph score.)

b. Let G be a graph with 9 vertices, each of degree 5 or 6. Prove that it has at least 5 vertices of degree 6 or at least 6 vertices of degree 5.

Problem 5

Show that if G is a connected undirected graph with k vertices of odd degree ($k > 0$), then there are $k/2$ walks, no two of which share an edge, that between them contain all the edges.