Computer Science 341
Discrete Mathematics

Homework 11
Due at noon on Fri, Dec 13, 2002

Collaboration Policy: You may collaborate in groups of at most 3 students. These groups must be disjoint and discussion across groups is not allowed. **Collaboration is limited to discussion of ideas only, and you should write up the solutions entirely on your own** and list your collaborators.

Problem 1

(a) Suppose we toss a fair coin until two consecutive heads (HH) come up. What is the expected number of tosses ? (Include the two tosses HH in your count of tosses).
(b) Suppose instead, we toss a fair coin until we see a head followed by a tail (HT). What is the expected number of tosses ? (Again, include the last two tosses HT in your count).
(c) Now, we toss a fair coin until we either see the pattern HH or HT for the first time. What is the probability that we see HH first ?

Problem 2

Let R be the number of runs in n independent tosses of a biased coin. (Runs are consecutive tosses with the same result). Each coin toss is a head (H) with probability p and is a tail (T) with probability $1-p$. Show that the variance of R is at most $4n \cdot p(1-p)$. (*Hint:* Let X_i be an indicator variable for the event that a new run begins at the ith toss. Express R in terms of X_i.) Note that the variance of a random variable R is $E[(R-\mu)^2]$ where $\mu = E[R]$.

Problem 3

Show that for $k > 1$, if $n < (\sqrt{k-1})2^{\frac{k}{2}}$, it is possible to divide the integers $\{1, 2, \ldots, n\}$ into two sets such that neither set contains an arithmetic progression of length k. (An arithmetic progression of length k is a subset of numbers of the form $a, a+d, a+2d, \ldots a+(k-1)d$.)

Problem 4

A grand party is thrown to celebrate the end of CS 341. As you leave the party, you walk n steps where each step is taken randomly to the left or the right each with probability $1/2$, independently of the other steps. (Despite your exhilarated state, you manage to walk in a straight line and each step is the same length).
(a) What is the expected distance (in terms of steps) from your initial location ? Find an expression for the expection (you can leave it as a summation), and show that this expectation is at most $c\sqrt{n}$ for some constant c.
(b) Show that the probability that you are more than $10\sqrt{n}$ steps away from your initial location is at most 0.01.