# COS 341 Discrete Mathematics

## Exponential Generating Functions and Recurrence Relations



## Textbook ?

- 12 students said they needed the textbook, but only one student bought the text from Triangle.
- If you don't have the book, please get it from Triangle.
- We will have to pay for the copies that aren't taken !



## Derangements (or Hatcheck lady revisited)

$d_n$ : number of permutations on $n$ objects without a fixed point

$D(x)$ : exponential generating function for number of derangements

A permutation on $[n]$ can be constructed by picking a subset $K$ of $[n]$, constructing a derangement of K and fixing the elements of $[n]-K$.

Every permutation of [n] arises exactly once this way.

$$\text{EGF for all permutations} = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \frac{1}{1-x}$$

$$\text{EGF for permutations with all elements fixed} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$$

$$\frac{1}{1-x} = D(x) \cdot e^x$$



## Derangements

$$\frac{1}{1-x} = D(x) \cdot e^x$$

$$D(x) = e^{-x} \frac{1}{1-x} = \left( \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \right) \left( \sum_{n=0}^{\infty} x^n \right)$$

$$\frac{d_n}{n!} = \text{coefficient of } x^n = \left( \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right)$$

$$d_n = n! \left( \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right)$$

Different proof in Matousek 10.2, problem 17

# Example

How many sequences of $n$ letters can be formed from A, B, and C such that the number of A's is odd and the number of B's is odd ?

$$\text{EGF for A's } = \sum_{n \text{ odd}} \frac{x^n}{n!} = \frac{e^x - e^{-x}}{2}$$

$$\text{EGF for B's } = \frac{e^x - e^{-x}}{2}$$

$$\text{EGF for C's } = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\text{required EGF} = \left(\frac{e^x - e^{-x}}{2}\right)^2 e^x = \frac{e^{3x} - 2e^x + e^{-x}}{4}$$



# Example

How many sequences of $n$ letters can be formed from A, B, and C such that the number of A's is odd and the number of B's is odd ?

$$\text{required EGF} = \frac{e^{3x} - 2e^x + e^{-x}}{4}$$

$$\text{coefficient of } x^n = \frac{1}{n!}\left(\frac{3^n - 2 + (-1)^n}{4}\right)$$

$$\text{required number} = \frac{3^n - 2 + (-1)^n}{4}$$



# Recurrence relations

A recurrence relation for the sequence $\{a_n\}$ is an equation that expressed $a_n$ in terms of one or more of the previous terms of the sequence, for all integers $n \geq n_0$

A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

$$a_n = a_{n-1} - a_{n-2} \qquad n \geq 2$$

$$\boxed{a_0 = 3, \quad a_1 = 5} \longleftarrow \text{Initial conditions}$$

$$a_2 = 5 - 3 = 2$$

$$a_3 = 2 - 5 = -3$$



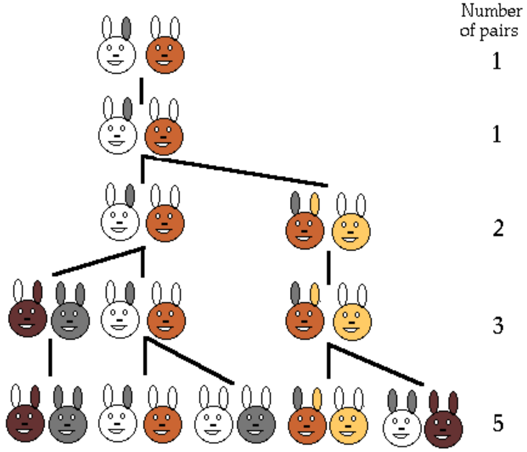
# Reproducing rabbits

Suppose a newly-born pair of rabbits, one male, one female, are placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are two months old, each pair of rabbits produces another pair each month.

Suppose that our rabbits **never die** and that the female **always** produces one new pair (one male, one female) **every month** from the second month on. The puzzle that Fibonacci posed was...

How many rabbits will there be in one year ?

## In the beginning …






## Enter recurrence relations

Let $f_n$ be the number of rabbits after $n$ months.

$$f_1 = 1, \quad f_2 = 1$$

$$f_n = f_{n-1} + f_{n-2}$$

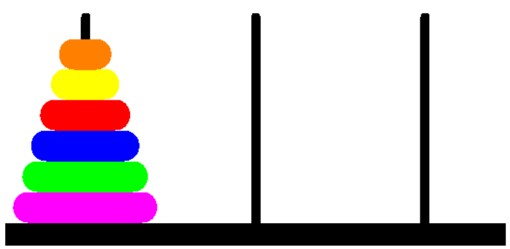
Number of pairs 2 months old

Number of existing pairs

Fibonacci sequence



## Towers of Hanoi



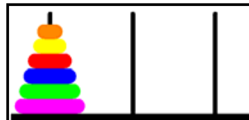
N disks are placed on first peg in order of size.

The goal is to move the disks from the first peg to the second peg

Only one disk can be moved at a time

A larger disk cannot be placed on top of a smaller disk





## Towers of Hanoi

Let $H_n$ be the number of moves required to solve the problem with $n$ disks.

$$H_1 = 1$$

$$H_n = 2H_{n-1} + 1$$

## Towers of Hanoi solution

$$
\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&\vdots \\
&\vdots \\
&= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\
&= 2^n - 1
\end{aligned}
$$



## The end of the world ?

An ancient legend says that there is a tower in Hanoi where the monks are transferring 64 gold disks from one peg to another according to the rules of the puzzle.

The monks work day and night, taking 1 second to transfer each disk. The myth says that the world will end when they finish the puzzle.

How long will this take ?

$$2^{64} - 1 = 18,446,744,073,709,551,615$$

It will take more than 500 billion years to solve the puzzle !



## More recurrence relations

How many bit strings of length n do not have 2 consecutive 0's ?

Let $a_n$ be the number of such bit strings of length $n$.

| | | | |
|---|---|---|---|
| End with 1: | Any bit string of length n-1 with no two consecutive 0's | 1 | $a_{n-1}$ |
| End with 0: | Any bit string of length n-2 with no two consecutive 0's | 1 0 | $a_{n-2}$ |

$$a_n = a_{n-1} + a_{n-2}$$



## A familiar sequence ?

How many bit strings of length n do not have 2 consecutive 0's ?

Let $a_n$ be the number of such bit strings of length $n$.

$$
\begin{aligned}
a_n &= a_{n-1} + a_{n-2} \\
a_1 &= 2 = f_3 \\
a_2 &= 3 = f_4 \\
a_3 &= 3 + 2 = 5 = f_5 \\
a_4 &= 5 + 3 = 8 = f_6 \\
a_n &= f_{n+2}
\end{aligned}
$$

## How many ways can you multiply ?

In how many ways can you parenthesize the product of $n+1$ numbers $x_0 \cdot x_1 \cdot x_2 \cdots x_n$ ?

e.g. $((x_0 \cdot x_1)\cdot x_2)\cdot x_3,$

$(x_0 \cdot (x_1 \cdot x_2)))\cdot x_3,$

$(x_0 \cdot x_1)\cdot (x_2 \cdot x_3),$

$x_0 \cdot ((x_1 \cdot x_2)\cdot x_3),$

$x_0 \cdot (x_1 \cdot (x_2 \cdot x_3)).$

Let $C_n$ be the number of ways of parenthesizing a product of $n+1$ numbers.

$C_3 = 5$



## How many ways can you multiply ?

Let $C_n$ be the number of ways of parenthesizing the product $x_0 \cdot x_1 \cdot x_2 \cdots x_n$

$$\underbrace{(x_0 \cdot x_1 \cdots x_k)}_{C_k}\cdot \underbrace{(x_{k+1} \cdots x_n)}_{C_{n-k-1}}$$

$$C_n = \sum_{k=0}^{n-1} C_k \cdot C_{n-k-1}$$

$n^{th}$ Catalan number

$C_0 = 1, \quad C_1 = 1$



## Solving recurrence relations

A linear homogeneous recurrence relation of degree $k$ with constant coefficients is a recurrence relation of the form

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$

where $c_1, \ldots c_k$ are real numbers and $c_k \neq 0$

$$a_n = a_{n-5}$$

$$a_n = a_{n-1} + a_{n-2}^2$$

$$a_n = n \cdot a_{n-1}$$



## Solving recurrence relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Try $a_n = r^n$

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

$$r^n - c_1 r^{n-1} - c_2 r^{n-2} - \cdots - c_k r^{n-k} = 0$$

$$\boxed{r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r^{k-1} - c_k = 0}$$

Characteristic equation

## Solving recurrence relations

$$a_2 = c_1 a_{n-1} + c_2 a_{n-2}$$

Let $r_1, r_2$ be the roots of the characteristic equation

$r^2 - c_1 r - c_2 = 0$

Then the solution to the recurrence relation is of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$



## Solving recurrence relations

$$f_n = f_{n-1} + f_{n-2}$$

Let $r_1, r_2$ be the roots of the characteristic equation

$r^2 - r - 1 = 0$

$r_1 = \left(1 + \sqrt{5}\right)/2$

$r_2 = \left(1 - \sqrt{5}\right)/2$

Then the solution to the recurrence relation is

of the form $f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$



## Solving recurrence relations

$$f_n = f_{n-1} + f_{n-2}$$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$