COS 341 Discrete Mathematics

Exponential Generating Functions

Trouble keeping pace ?

- Read the textbook
- Optional reference text (Rosen) has many more solved exercises and practice problems
- Start early on homework assignments
- Come to office hours, make separate appointments
- Learn from discussions with fellow students

- Tutoring:
- Seniors: See Dean Richard Williams (408 West College, 8-5520)
- Juniors: See Dean Frank Ordiway (404 West College, 8-1998)
- Sophomores: See Director of Studies in your home college

Ordinary Generating Functions

$(a_0, a_1, a_2, \ldots)$: sequence of real numbers

Ordinary Generating Function of this sequence is the power series $a(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$

Exponential Generating Functions

$(a_0, a_1, a_2, \ldots)$: sequence of real numbers

Exponential Generating function of this sequence is the power series

$$a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot x^i$$

Exponential generating function examples

What is the generating function for the sequence $(1,1,1,1,\ldots)$?

$$\sum_{i=0}^{\infty}\frac{1}{i!}x^i = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = e^x$$

What is the generating function for the sequence $(1,2,4,8,\ldots)$?

$$\sum_{i=0}^{\infty}\frac{2^i}{i!}x^i = 1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots = e^{2x}$$

Operations on exponential generating functions

- Addition

  $(a_0+b_0, a_1+b_1, \ldots)$ has generating function $a(x)+b(x)$
- Multiplication by fixed real number

  $(\alpha a_0, \alpha a_1, \ldots)$ has generating function $\alpha a(x)$
- Shifting the sequence to the right

  $\cancel{(\underbrace{0,\ldots 0}_{n\times}, a_0, a_1, \ldots)}$ ~~has generating function~~ $\cancel{x^n a(x)}$
- Shifting to the left

  $\cancel{(a_k, a_{k+1}, \ldots)}$ ~~has generating function~~ $\cancel{\frac{a(x)-\sum_{i=0}^{k-1} a_i \cdot x^i}{x^n}}$

- Substituting αx for x

  $(a_0, \alpha a_1, \alpha^2 a_2 \ldots)$ has generating function $a(\alpha x)$
- Substitute x^n for x

  $\cancel{(a_0, \underbrace{0,\ldots 0}_{n-1\times}, a_1, \underbrace{0,\ldots 0}_{n-1\times}, a_2 \ldots)}$ ~~has generating function~~ $\cancel{a(x^n)}$

- Differentiation

  $\cancel{(a_1, 2a_2, 3a_3 \ldots)}$ ~~has generating function~~ $\cancel{\frac{d}{dx}a(x)}$ (or $a'(x)$)
- Integration

  $\cancel{(0, a_0, \tfrac{1}{2}a_1, \tfrac{1}{3}a_2 \ldots)}$ ~~has generating function~~ $\cancel{\int_0^x f(t)dt}$
- Multiplication of generating functions

  $$\cancel{\left(\sum_{n=0}^{\infty} a_n \cdot x^n\right)\left(\sum_{n=0}^{\infty} b_n \cdot x^n\right) = \left(\sum_{n=0}^{\infty} c_n \cdot x^n\right)}$$

  $$\cancel{c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}}$$

Differentiation

$a(x)$ is the exponential generating function for $(a_0, a_1, a_2, \ldots)$

$$a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot x^i$$

$$\frac{d}{dx} a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot i \cdot x^{i-1} = \sum_{i=1}^{\infty} \frac{a_i}{(i-1)!} \cdot x^{i-1}$$

$\frac{d}{dx} a(x)$ is the exponential generating function for $(a_1, a_2, a_3, \ldots)$

Differentiation is equivalent to shifting the sequence to the left

Integration

$a(x)$ is the exponential generating function for $(a_0, a_1, a_2, \ldots)$

$$a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot x^i$$

$$\int_0^x a(t)dt = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot \int_0^x t^i dt = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot \frac{x^{i+1}}{(i+1)} = \sum_{i=0}^{\infty} \frac{a_i}{(i+1)!} \cdot x^{i+1}$$

$\int_0^x a(t)dt$ is the exponential generating function for $(0, a_0, a_1, a_2, \ldots)$

Integration is equivalent to shifting the sequence to the right

Multiplication

Ordinary $\left(\sum_{n=0}^{\infty} a_n \cdot x^n\right)\left(\sum_{n=0}^{\infty} b_n \cdot x^n\right) = \left(\sum_{n=0}^{\infty} c_n \cdot x^n\right)$

$$c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}$$

Exponential $\left(\sum_{n=0}^{\infty} \frac{a_n}{n!} \cdot x^n\right)\left(\sum_{n=0}^{\infty} \frac{b_n}{n!} \cdot x^n\right) = \left(\sum_{n=0}^{\infty} \frac{c_n}{n!} \cdot x^n\right)$

$$\frac{c_n}{n!} = \sum_{k=0}^{n} \frac{a_k}{k!} \cdot \frac{b_{n-k}}{(n-k)!}$$

$$c_n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a_k \cdot b_{n-k} = \sum_{k=0}^{n} \binom{n}{k} a_k \cdot b_{n-k}$$

Implications of product rule

$$C(x) = A(x)B(x)$$

Ordinary $\boxed{c_n = \sum_{k=0}^{n} a_k \cdot b_{n-k}}$

Useful for counting with indistinguishable objects

Exponential $\boxed{c_n = \sum_{k=0}^{n} \binom{n}{k} a_k \cdot b_{n-k}}$

Useful for counting with ordered objects

Interpretation of Multiplication: Product Rule

Given arrangements of type A and type B, define arrangements of type C for n labeled objects as follows:

Divide the group of n labeled objects into two groups, the First group and the Second group;
arrange the First group by an arrangement of type A and the Second group by an arrangement of type B.

a_n : number of arrangements of type A for n objects
b_n : number of arrangements of type B for n objects
c_n : number of arrangements of type C for n objects

Interpretation of Multiplication: Product Rule

a_n : number of arrangements of type A for n people
b_n : number of arrangements of type B for n people
c_n : number of arrangements of type C for n people

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k \cdot b_{n-k}$$

a_n : exponential generating function $A(x)$
b_n : exponential generating function $B(x)$
c_n : exponential generating function $C(x)$

$$C(x) = A(x)B(x)$$

$A(x)$: exponential generating function for arrangements of type A
$a_0 = 0$: no empty group allowed

Define arrangements of type D for n labeled objects as follows:

Divide the group of n labeled objects into k groups,
the First group, Second group,…, kth group $(k = 0, 1, 2, \ldots)$
arrange each group by an arrangement of type A .

$D(x)$: exponential generating function for arrangements of type D

$D_k(x)$: exponential generating function for arrangements of type D with exactly k groups

$$D_k(x) = A(x)^k$$

$$D(x) = \sum_{k=0}^{\infty} D_k(x) = \sum_{k=0}^{\infty} A(x)^k = \frac{1}{1 - A(x)}$$

$A(x)$: exponential generating function for arrangements of type A

$a_0 = 0$: no empty group allowed

Define arrangements of type E for n labeled objects as follows:

Divide the group of n labeled objects into k groups,
and arrange each group by an arrangement of type A
(the groups are not numbered) .

$E(x)$: exponential generating function for arrangements of type E

$E_k(x)$: exponential generating function for arrangements of type E
with exactly k groups

$$E_k(x) = \frac{A(x)^k}{k!}$$
$$E(x) = \sum_{k=0}^{\infty} E_k(x)$$
$$= \sum_{k=0}^{\infty} \frac{A(x)^k}{k!}$$
$$= e^{A(x)}$$

Example

How many ways can n people be arranged into pairs,
(the pairs are not numbered) ?

$A(x)$: exponential generating function for a single pair

$$a_2 = 1,\ a_i = 0 \text{ for } i \neq 2$$
$$A(x) = \frac{x^2}{2}$$

$E(x)$: exponential generating function for arranging
n people into pairs

$$E(x) = e^{\frac{x^2}{2}}$$

$$x^n \text{ term} = \frac{e_n}{n!} x^n = \left(\frac{x^2}{2}\right)^{n/2} \frac{1}{(n/2)!} \qquad e_n = \frac{n!}{2^{n/2}(n/2)!}$$

Derangements (or Hatcheck lady revisited)

d_n : number of permutations on n objects without a fixed point

$D(x)$: exponential generating function for number of derangements

A permutation on $[n]$ can be constructed by picking
a subset K of $[n]$, constructing a derangement of K and
fixing the elements of $[n]$ - K.
Every permutation of [n] arises exactly once this way.

$$\text{EGF for all permutations } = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \frac{1}{1-x}$$

$$\text{EGF for permutations with all elements fixed } = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$$

$$\frac{1}{1-x} = D(x) \cdot e^x$$

Derangements

$$\frac{1}{1-x} = D(x)\cdot e^x$$

$$D(x) = e^{-x}\frac{1}{1-x}$$

$$= \left(\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}x^n\right)$$

$$\frac{d_n}{n!} = \text{coefficient of } x^n = \left(\sum_{k=0}^{n}\frac{(-1)^k}{k!}\right)$$

$$d_n = n!\left(\sum_{k=0}^{n}\frac{(-1)^k}{k!}\right)$$

Different proof in Matousek 10.2, problem 17

Example

How many sequences of n letters can be formed from A, B, and C such that the number of A's is odd and the number of B's is odd ?

$$\text{EGF for A's} = \sum_{n\text{ odd}}\frac{x^n}{n!} = \frac{e^x - e^{-x}}{2}$$

$$\text{EGF for B's} = \frac{e^x - e^{-x}}{2}$$

$$\text{EGF for C's} = \sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x$$

$$\text{required EGF} = \left(\frac{e^x - e^{-x}}{2}\right)^2 e^x = \frac{e^{3x} - 2e^x + e^{-x}}{4}$$

Example

How many sequences of n letters can be formed from A, B, and C such that the number of A's is odd and the number of B's is odd ?

$$\text{required EGF} = \frac{e^{3x} - 2e^x + e^{-x}}{4}$$

$$\text{coefficient of } x^n = \frac{1}{n!}\left(\frac{3^n - 2 + (-1)^n}{4}\right)$$

$$\text{required number} = \frac{3^n - 2 + (-1)^n}{4}$$