

Ordinary Generating Functions

 $(a_0, a_1, a_2, ...)$: sequence of real numbers

Ordinary

Generating Function of this sequence is the power series $a(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$

Trouble keeping pace ?

- Read the textbook
- Optional reference text (Rosen) has many more solved exercises and practice problems
- Start early on homework assignments
- Come to office hours, make separate appointments
- Learn from discussions with fellow students
- Tutoring:
- Seniors: See Dean Richard Williams (408 West College, 8-5520)
- Juniors: See Dean Frank Ordiway (404 West College, 8-1998)
- Sophomores: See Director of Studies in your home college

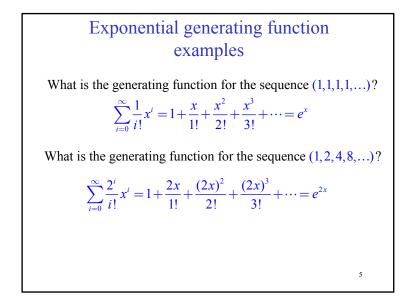
Exponential Generating Functions

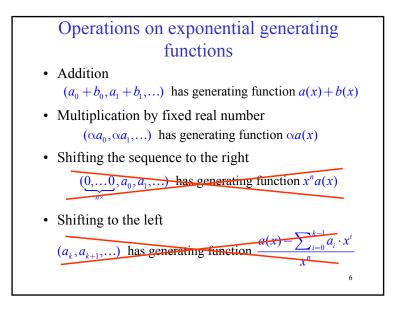
 $(a_0, a_1, a_2, ...)$: sequence of real numbers Exponential Generating function of this sequence is the power series

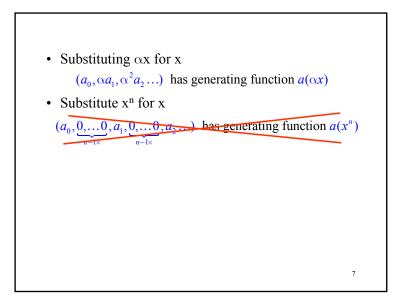
$$a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot x^i$$

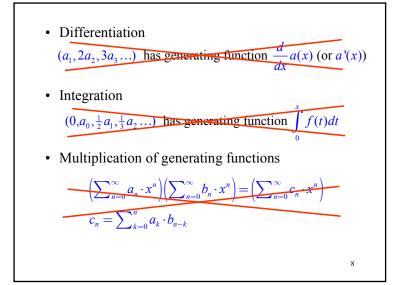
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Differentiation

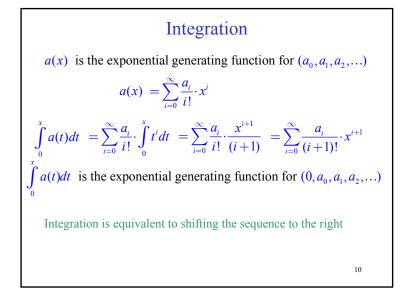
a(x) is the exponential generating function for $(a_0, a_1, a_2, ...)$

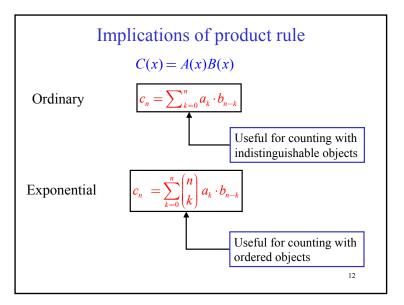
$$a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot x^i$$
$$\frac{d}{dx}a(x) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \cdot i \cdot x^{i-1} = \sum_{i=1}^{\infty} \frac{a_i}{(i-1)!} \cdot x^{i-1}$$

 $\frac{d}{dx}a(x)$ is the exponential generating function for $(a_1, a_2, a_3, ...)$

Differentiation is equivalent to shifting the sequence to the left

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Interpretation of Multiplication: Product Rule

Given arrangements of type A and type B, define arrangements of type C for n labeled objects as follows:

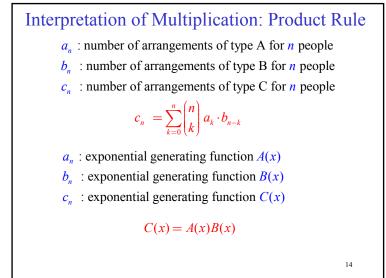
Divide the group of n labeled objects into two groups, the First group and the Second group; arrange the First group by an arrangement of type A and the Second group by an arrangement of type B.

 a_n : number of arrangements of type A for *n* objects

- b_n : number of arrangements of type B for *n* objects
- c_n : number of arrangements of type C for *n* objects

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A(x): exponential generating function for arrangements of type A $a_0 = 0$: no empty group allowed

Define arrangements of type D for n labeled objects as follows:

Divide the group of *n* labeled objects into *k* groups, the First group, Second group,..., *k*th group (k = 0, 1, 2, ...) arrange each group by an arrangement of type A.

D(x): exponential generating function for arrangements of type D

 $D_k(x)$: exponential generating function for arrangements of type D with exactly k groups

$$D_k(x) = A(x)^k$$
$$D(x) = \sum_{k=0}^{\infty} D_k(x)$$
$$= \sum_{k=0}^{\infty} A(x)^k$$
$$= \frac{1}{1 - A(x)}$$

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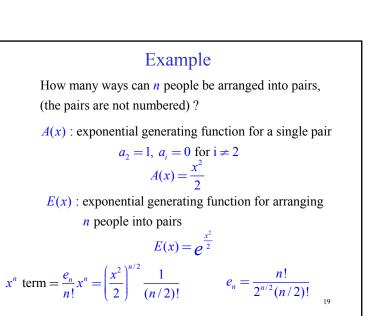
A(x): exponential generating function for arrangements of type A $a_0 = 0$: no empty group allowed

Define arrangements of type E for n labeled objects as follows:

Divide the group of n labeled objects into k groups, and arrange each group by an arrangement of type A (the groups are not numbered).

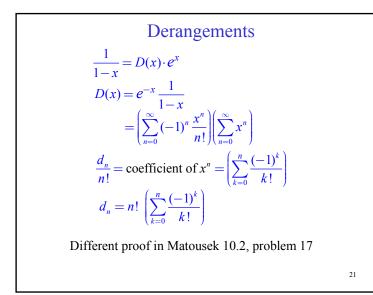
E(x): exponential generating function for arrangements of type E

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$E_{k}(x) : \text{ exponential generating function for arrangements of ty}$ with exactly k groups $E_{k}(x) = \frac{A(x)^{k}}{k!}$ $E(x) = \sum_{k=0}^{\infty} E_{k}(x)$ $= \sum_{k=0}^{\infty} \frac{A(x)^{k}}{k!}$	уре E
$-\sum_{k=0}^{k=0}A(x)^{k}$	
$= \sum_{k=0}^{\infty} \frac{k!}{k!}$ $= e^{A(x)}$	
$=e^{A(\lambda)}$	
18	8

Derangements (or Hatcheck lady revisited) d_n : number of permutations on n objects without a fixed point D(x): exponential generating function for number of derangements A permutation on [n] can be constructed by picking a subset K of [n], constructing a derangement of K and fixing the elements of [n] - K. Every permutation of [n] arises exactly once this way. EGF for all permutations $= \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \frac{1}{1-x}$ EGF for permutations with all elements fixed $= \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$ $\frac{1}{1-x} = D(x) \cdot e^x$



Example How many sequences of *n* letters can be formed from A, B, and C such that the number of A's is odd and the number of B's is odd ? $required EGF = \frac{a^3 - 2e^x + e^{-x}}{4}$ $coefficient of x^n = \frac{1}{n!} \left(\frac{a^n - 2 + (-1)^n}{4} \right)$ $required number = \frac{a^n - 2 + (-1)^n}{4}$

Example

How many sequences of n letters can be formed from A, B, and C such that the number of A's is odd and the number of B's is odd ?

