

COS 341 Discrete Mathematics

Generating Functions

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Administrative Issues

- Homework 1 has been graded
- Median score: 77

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Generating functions

(a_0, a_1, a_2, \dots) : sequence of real numbers

Generating function of this sequence is

the power series $a(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$

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Operations on power series

- Addition
 $(a_0 + b_0, a_1 + b_1, \dots)$ has generating function $a(x) + b(x)$
- Multiplication by fixed real number
 $(\alpha a_0, \alpha a_1, \dots)$ has generating function $\alpha a(x)$
- Shifting the sequence to the right
 $(\underbrace{0, \dots, 0}_{n \times}, a_0, a_1, \dots)$ has generating function $x^n a(x)$
- Shifting to the left
 (a_k, a_{k+1}, \dots) has generating function $\frac{a(x) - \sum_{i=0}^{k-1} a_i \cdot x^i}{x^n}$

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- Substituting αx for x

$(a_0, \alpha a_1, \alpha^2 a_2 \dots)$ has generating function $a(\alpha x)$

- Substitute x^n for x

$(a_0, \underbrace{0, \dots, 0}_{n-1x}, a_1, \underbrace{0, \dots, 0}_{n-1x}, a_2 \dots)$ has generating function $a(x^n)$

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- Differentiation

$(a_1, 2a_2, 3a_3 \dots)$ has generating function $\frac{d}{dx} a(x)$ (or $a'(x)$)

- Integration

$(0, a_0, \frac{1}{2}a_1, \frac{1}{3}a_2 \dots)$ has generating function $\int_0^x f(t) dt$

- Multiplication of generating functions

$$\left(\sum_{n=0}^{\infty} a_n \cdot x^n\right) \left(\sum_{n=0}^{\infty} b_n \cdot x^n\right) = \left(\sum_{n=0}^{\infty} c_n \cdot x^n\right)$$

$$c_n = \sum_{k=0}^n a_k \cdot b_{n-k}$$

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Applying the toolkit

What is the generating function for the sequence $(1^2, 2^2, 3^2, \dots)$

$$a_k = (k+1)^2$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx} \left(\frac{1}{1-x} \right) = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$\frac{2}{(1-x)^3} = \frac{d}{dx} \left(\frac{1}{(1-x)^2} \right) = 1 \cdot 2 + 3 \cdot 2x + 4 \cdot 3x^2 + 5 \cdot 4x^3 + \dots$$

$$\frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} = 1 \cdot 1 + 2 \cdot 2x + 3 \cdot 3x^2 + 4 \cdot 4x^3 + \dots$$

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An alternate derivation: Generalized Binomial Theorem

$$(1+x)^r = \binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \dots$$

$$\binom{r}{k} = \frac{r(r-1)(r-2)\dots(r-k+1)}{k!}$$

$$\binom{-n}{k} = \frac{-n(-n-1)(-n-2)\dots(-n-k+1)}{k!}$$

$$= (-1)^k \frac{n(n+1)(n+2)\dots(n+k-1)}{k!}$$

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k} = (-1)^k \binom{n+k-1}{n-1}$$

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An alternate derivation:

Generalized Binomial Theorem

$$(1+x)^r = \binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \dots$$

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k} = (-1)^k \binom{n+k-1}{n-1}$$

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{n-1} x^k$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{n-1} (-x)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k$$

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An alternate derivation:

Generalized Binomial Theorem

$$(1-x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{n-1} (-x)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k$$

$$(1-x)^{-1} = \sum_{k=0}^{\infty} \binom{k}{0} x^k = \sum_{k=0}^{\infty} x^k$$

$$(1-x)^{-2} = \sum_{k=0}^{\infty} \binom{k+1}{1} x^k = \sum_{k=0}^{\infty} (k+1)x^k$$

$$(1-x)^{-3} = \sum_{k=0}^{\infty} \binom{k+2}{2} x^k = \sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2} x^k$$

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An alternate derivation:

Generalized Binomial Theorem

$$(1-x)^{-2} = \sum_{k=0}^{\infty} \binom{k+1}{1} x^k = \sum_{k=0}^{\infty} (k+1)x^k$$

$$(1-x)^{-3} = \sum_{k=0}^{\infty} \binom{k+2}{2} x^k = \sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2} x^k$$

$$2(1-x)^{-3} - (1-x)^{-2} = \sum_{k=0}^{\infty} (k+1)(k+1)x^k$$

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More toolkit examples

What is the generating function of the sequence

$(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$?

$$-\frac{\ln(1-x)}{x} = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \dots$$

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More toolkit examples

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\int_0^x \frac{dt}{1-t} = \int_0^x (1+t+t^2+t^3+t^4+\dots)dt$$

$$-\ln(1-x) + \ln(1) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots$$

$$\frac{-\ln(1-x)}{x} = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \dots$$

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Applications to counting

A box contains 30 red, 40 blue and 50 green balls.
Balls of the same color are indistinguishable.

How many ways are there of selecting a collection of 70 balls from the box ?

coefficient of x^{70} in

$$(1 + x + x^2 + \dots + x^{30})$$

$$\times (1 + x + x^2 + \dots + x^{40})$$

$$\times (1 + x + x^2 + \dots + x^{50})$$

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Enter generating functions

coefficient of x^{70} in

$$(1 + x + x^2 + \dots + x^{30})(1 + x + x^2 + \dots + x^{40})(1 + x + x^2 + \dots + x^{50})$$

$$(1 + x + x^2 + \dots + x^{30}) = \frac{1 - x^{31}}{1 - x}$$

Sum of first n terms of a geometric series

Alternately $\frac{1}{1-x} = 1 + x + x^2 + \dots$

$$\frac{x^{31}}{1-x} = x^{31} + x^{32} + x^{33} + \dots$$

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Enter generating functions

coefficient of x^{70} in

$$(1 + x + x^2 + \dots + x^{30})(1 + x + x^2 + \dots + x^{40})(1 + x + x^2 + \dots + x^{50})$$

$$\text{coefficient of } x^{70} \text{ in } \frac{(1-x^{31})}{1-x} \frac{(1-x^{41})}{1-x} \frac{(1-x^{51})}{1-x}$$

$$\frac{1}{(1-x)^3} (1-x^{31})(1-x^{41})(1-x^{51})$$

$$= \left(\sum_{k=0}^{\infty} \binom{k+2}{2} x^k \right) (1-x^{31} - x^{41} - x^{51} + \dots)$$

$$\binom{70+2}{2} - \binom{70-31+2}{2} - \binom{70-41+2}{2} - \binom{70-51+2}{2} = 1061$$

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More tricks with generating functions

$$a(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$$

$$b_n = \sum_{i=0}^n a_i$$

What is $b(x) = \sum_{i=0}^{\infty} b_i \cdot x^i$

$$b(x) = \frac{a(x)}{1-x}$$

$$\sum_{i=0}^{\infty} b_i \cdot x^i = (a_0 + a_1x + a_2x^2 + \dots)(1 + x + x^2 + \dots)$$

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More tricks with generating functions

What is $(1^2 + 2^2 + \dots + n^2)$?

$$\frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} = 1 \cdot 1 + 2 \cdot 2x + 3 \cdot 3x^2 + 4 \cdot 4x^3 + \dots$$

$$b_n = \sum_{i=0}^n a_i \quad b(x) = \frac{a(x)}{1-x}$$

$$b_0 = 1^2$$

$$b_1 = 1^2 + 2^2$$

$$b_2 = 1^2 + 2^2 + 3^2$$

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More tricks with generating functions

What is $(1^2 + 2^2 + \dots + n^2)$?

$$\frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} = 1 \cdot 1 + 2 \cdot 2x + 3 \cdot 3x^2 + 4 \cdot 4x^3 + \dots$$

$$\frac{2}{(1-x)^4} - \frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} b_n x^n$$

$$b_n = 1^2 + 2^2 + \dots + (n+1)^2$$

$$b_n = 2 \binom{3+n}{3} - \binom{2+n}{2}$$

$$b_{n-1} = 2 \binom{2+n}{3} - \binom{1+n}{2}$$

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More tricks with generating functions

What is $(1^2 + 2^2 + \dots + n^2)$?

$$b_{n-1} = 1^2 + 2^2 + \dots + n^2$$

$$= 2 \binom{2+n}{3} - \binom{1+n}{2}$$

$$= \frac{2(n+2)(n+1)n}{6} - \frac{n(n+1)}{2}$$

$$= \frac{(2n+1)(n+1)n}{6}$$

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More tricks with generating functions

What is $\sum_{k=0}^m (-1)^k \binom{n}{k}$?

The generating function for the sequence

$$a_k = (-1)^k \binom{n}{k} \text{ is } a(x) = (1-x)^n$$

The generating function for the sequence

$$c_m = \sum_{k=0}^m a_k \text{ is } \frac{a(x)}{1-x} = (1-x)^{n-1}$$

$$c_m = \text{coefficient of } x^m = (-1)^m \binom{n-1}{m}$$