COS 341   Discrete Mathematics

Advanced Counting
Administrative Issues

- **Textbook update:**
  - Bookstore will get one additional copy of textbook.
  - Copies will be available at **Triangle, 150 Nassau Street (Tel: 609 924 4630)** for $35 beginning today. They are open M-F 8am-6pm.

- **Updated collaboration policy**

- **Tutoring ?**

- **Readings for this week:** Matousek and Nesetril, Chapter 10
Inclusion-Exclusion principle

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]
Inclusion-Exclusion principle

\[ |A_1 \cup A_2 \cup \cdots \cup A_n| \]

\[ = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i_1 \leq i_2 \leq n} |A_{i_1} \cap A_{i_2}| + \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n| \]
Hatcheck lady problem

n gentlemen arrive at a party and leave their hats in the cloak room. On their departure, the hatcheck lady absent-mindedly hands back a hat to each man at random.

What is the probability that none of the men receives their own hat?

- $n!$ ways of assigning hats back to men
- What fraction of these assignments are such that no man receives his own hat?
Hatcheck lady

- Number hats and men 1,2,..,n
- $\pi(i)$: number of hat received by $i$th man
- $\pi$ is a permutation
- Index $i$ with $\pi(i) = i$ is a fixed point of $\pi$
- $D(n)$: number of permutations with no fixed point
Enter inclusion-exclusion

- \( S_n \): set of all permutations
- \( A_i = \{ \pi \in S_n : \pi(i) = i \} \)
- \( \bigcup_i A_i : \) bad permutations

\[
| A_i | = (n - 1)!
\]

\[
| A_i \cap A_j | = (n - 1)!
\]

\[
| A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} | = (n - k)!
\]
Enter inclusion-exclusion

\[ |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n - k)! \]

\[ |A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k}(n - k)! \]

\[ = \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!} \]

\[ D(n) = n! - |A_1 \cup A_2 \cup \cdots \cup A_n| \]

\[ = n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!} \]

\[ = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right) \]
Finishing up

\[ 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} \] converges to \( e^{-1} \)

\[ D(n) \approx \frac{n!}{e} \]

Probability that nobody gets their hat back converges to the constant \( e^{-1} = 0.36787 \) independent of the number of men!
Algebraic derivation of identities on binomial coefficients

Binomial Theorem:

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]

\[= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n\]

Equality of two polynomials implies equality of corresponding coefficients
Algebraic derivation of identities on binomial coefficients

\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}
\]

\[(1 + x)^n (1 + x)^n = (1 + x)^{2n}\]

coefficient of \(x^n\) on LHS

\[
\begin{align*}
&= \left(\binom{n}{0} \binom{n}{n}\right) + \left(\binom{n}{1} \binom{n}{n-1}\right) + \left(\binom{n}{2} \binom{n}{n-2}\right) + \cdots + \left(\binom{n}{n} \binom{n}{0}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}
\end{align*}
\]
Algebraic derivation of identities on binomial coefficients

\[
\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}
\]

\[(1 + x)^n (1 + x)^n = (1 + x)^{2n}\]

Coefficient of \(x^n\) on LHS
\[
= \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}
\]

Coefficient of \(x^n\) on RHS
\[
= \sum_{k=0}^{n} \binom{2n}{n}
\]
Algebraic derivation of identities on binomial coefficients

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n}{n-k} = ?
\]

\[
(1 - x)^n (1 + x)^n = (1 - x^2)^n
\]

Coefficient of \(x^n\) on LHS

\[
= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n}{n-k}
\]

Coefficient of \(x^n\) on RHS

\[
= \begin{cases} 
0 & \text{when } n \text{ odd} \\
(-1)^{n/2} \binom{n}{n/2} & \text{n even}
\end{cases}
\]
Algebraic derivation of identities on binomial coefficients

\[ \sum_{k=0}^{n} (-1)^k k \binom{n}{k} = ? \]

\[ (1 - x)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k \]

\[ \frac{d}{dx} (1 - x)^n = \frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k \]

\[ -n(1 - x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k \cdot x^{k-1} \]
Algebraic derivation of identities on binomial coefficients

\[ \sum_{k=0}^{n} (-1)^k k \binom{n}{k} = ? \]

\[ -n(1 - x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k \cdot x^{k-1} \]

substitute \( x = 1 \)

\[ 0 = \sum_{k=0}^{n} (-1)^k k \binom{n}{k} \]
**Power series**

Infinite series of the form \( a_0 + a_1x + a_2x^2 + \cdots \)

\[
\frac{1}{1-x} = 1 + x + x^2 + \cdots
\]

Series converges for \( x \) in the interval \((-1,1)\)

Function contains all the information about series

Differentiate \( k \) times and substitute \( x=0 \), we get \( k! \) times coefficient of \( x^k \)

Taylor series of the function \( \frac{1}{1-x} \) at \( x = 0 \)
Power series

\( (a_0, a_1, a_2, \ldots) : \) sequence of real numbers

\[ |a_n| \leq K^n \]

For any number \( x \in (-\frac{1}{K}, \frac{1}{K}) \), the series

\[ a(x) = \sum_{i=0}^{\infty} a_i \cdot x^i \]

converges

Values of \( a(x) \) in arbitrarily small neighborhood of 0 uniquely determine \( (a_0, a_1, a_2, \ldots) \)

\[ a_n = \frac{a^{(n)}(0)}{n!} \]
Generating functions

\((a_0, a_1, a_2, \ldots)\) : sequence of real numbers

Generating function of this sequence is the power series

\[ a(x) = \sum_{i=0}^{\infty} a_i \cdot x^i \]
Generating function basics

What is the generating function of the sequence \((1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)\)?

\[
1 + \frac{1}{2} x + \frac{1}{3} x^2 + \frac{1}{4} x^3 + \cdots
\]

\[
\frac{\ln(1 - x)}{x}
\]

\[
x + \frac{1}{2} x + \frac{1}{3} x^2 + \frac{1}{4} x^3 + \cdots = -\ln(1 - x)
\]

\[
1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots = e^x
\]
Generating function toolkit: Generalized binomial theorem

\[ \binom{r}{k} = \frac{r(r-1)(r-2)\ldots(r-k+1)}{k!} \]

\((1 + x)^r\) is the generating function for the sequence \(\left( \binom{r}{0}, \binom{r}{1}, \binom{r}{2}, \binom{r}{3}, \ldots \right)\)

The power series \(\binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \ldots\)

always converges for all \(|x| < 1\)
Negative binomial coefficients?

\[
\binom{r}{k} = (-1)^k \binom{-r + k - 1}{k} = (-1)^k \binom{-r + k - 1}{-r - 1}
\]

\[
\frac{1}{(1-x)^n} = \binom{n-1}{n-1} + \binom{n}{n-1} x + \binom{n+1}{n-1} x^2 + \cdots + \binom{n+k-1}{n-1} x^k + \cdots
\]

\[
\frac{1}{1-x} = 1 + x + x^2 + \cdots
\]
Operations on power series

• Addition
  \((a_0 + b_0, a_1 + b_1, \ldots)\) has generating function \(a(x) + b(x)\)

• Multiplication by fixed real number
  \((\alpha a_0, \alpha a_1, \ldots)\) has generating function \(\alpha a(x)\)

• Shifting the sequence
  \((0, \ldots 0, a_0, a_1, \ldots)\) has generating function \(x^n a(x)\)

• Shifting to the left
• Substituting $\alpha x$ for $x$

$$(a_0, \alpha a_1, \alpha^2 a_2 \ldots)$$ has generating function $a(\alpha x)$

$(1, 2, 4, 8, \ldots)$ has generating function?

• Substitute $x^n$ for $x$

$$(1, 1, 2, 2, 4, 4, 8, 8, \ldots)$$ has generating function?

$$\frac{1}{1 - 2x^2} + \frac{x}{1 - 2x^2}$$
• Integration and differentiation

\[(a_0, 2a_1, 3a_2 \ldots)\] has generating function?
\[(0, a_0, \frac{1}{2} a_1, \frac{1}{3} a_2 \ldots)\] has generating function?

• Multiplication of generating functions