COS 341 Discrete Mathematics

Advanced Counting

Administrative Issues

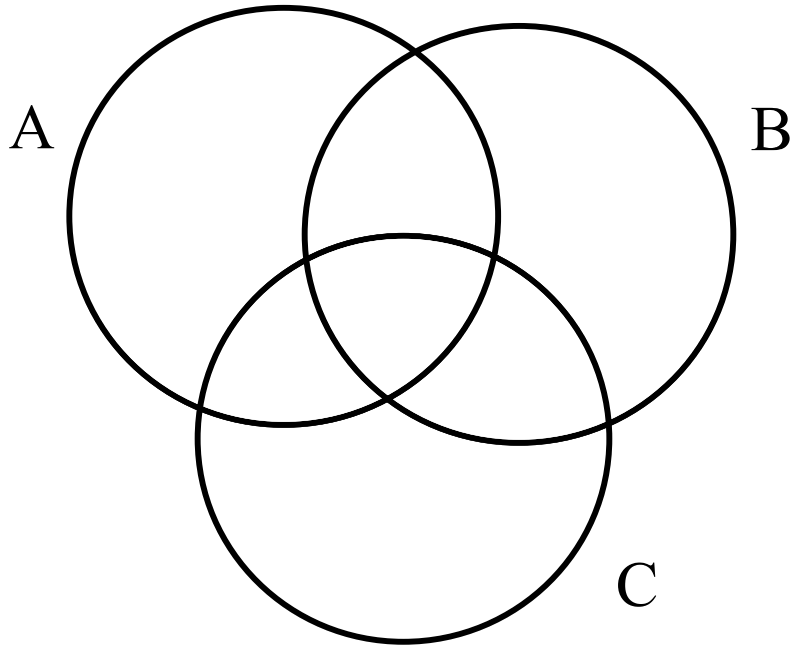
- **Textbook update:**
- Bookstore will get **one** additional copy of textbook.
- Copies will be available at **Triangle, 150 Nassau Street (Tel: 609 924 4630)** for $35 beginning today. They are open M-F 8am-6pm.

- **Updated collaboration policy**

- Tutoring ?

- Readings for this week: Matousek and Nesetril, Chapter 10

Inclusion-Exclusion principle



$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

Inclusion-Exclusion principle

$$| A_1 \cup A_2 \cup \cdots \cup A_n |$$

$$= \sum_{i=1}^{n} | A_i | - \sum_{1 \leq i_1 \leq i_2 \leq n}^{n} | A_{i_1} \cap A_{i_2} |$$

$$+ \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq n}^{n} | A_{i_1} \cap A_{i_2} \cap A_{i_3} |$$

$$- \cdots + (-1)^{n-1} | A_1 \cap A_2 \cap \cdots \cap A_n |$$

Hatcheck lady problem

n gentlemen arrive at a party and leave their hats in the cloak room. On their departure, the hatcheck lady absent-mindedly hands back a hat to each man at random.

What is the probability that none of the men receives their own hat ?

- n! ways of assigning hats back to men
- What fraction of these assignments are such that no man receives his own hat ?

Hatcheck lady

- Number hats and men 1,2,..,n
- $\pi(i)$: number of hat received by *i*th man
- π is a permutation
- Index i with $\pi(i) = i$ is a fixed point of π
- D(n): number of permutations with no fixed point

Enter inclusion-exclusion

- S_n: set of all permutations
- $A_i = \{\pi \in S_n : \pi(i) = i\}$
- $\cup_i A_i$: *bad* permutations

$$|A_i| = (n-1)!$$

$$|A_i \cap A_j| = (n-1)!$$

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!$$

Enter inclusion-exclusion

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!$$

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)!$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!}$$

$$D(n) = n! - |A_1 \cup A_2 \cup \cdots \cup A_n|$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!}$$

$$= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right)$$

Finishing up

$$1-\frac{1}{1!}+\frac{1}{2!}-\cdots+(-1)^n\frac{1}{n!} \quad \text{converges to } e^{-1}$$

$$D(n)\approx\frac{n!}{e}$$

Probability that nobody gets their hat back converges to the constant $e^{-1} = 0.36787$ independent of the number of men !

Algebraic derivation of identities on binomial coefficients

Binomial Theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n$$

Equality of two polynomials implies equality of corresponding coefficients

Algebraic derivation of identities on binomial coefficients

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(1+x)^n (1+x)^n = (1+x)^{2n}$$

coefficient of x^n on LHS

$$= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}$$

$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

Algebraic derivation of identities on binomial coefficients

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(1+x)^n (1+x)^n = (1+x)^{2n}$$

coefficient of x^n on LHS $= \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}$

coefficient of x^n on RHS $= \sum_{k=0}^{n} \binom{2n}{n}$

Algebraic derivation of identities on binomial coefficients

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n}{n-k} = ?$$

$$(1-x)^n (1+x)^n = (1-x^2)^n$$

coefficient of x^n on LHS $= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n}{n-k}$

coefficient of x^n on RHS $= \begin{cases} 0 & \text{when n odd} \\ (-1)^{n/2} \binom{n}{n/2} & \text{n even} \end{cases}$

Algebraic derivation of identities on binomial coefficients

$$\sum_{k=0}^{n} (-1)^k k \binom{n}{k} = ?$$

$$(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k$$

$$\frac{d}{dx}(1-x)^n = \frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k$$

$$-n(1-x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k \cdot x^{k-1}$$

Algebraic derivation of identities on binomial coefficients

$$\sum_{k=0}^{n}(-1)^k k\binom{n}{k} = ?$$

$$-n(1-x)^{n-1} = \sum_{k=0}^{n}\binom{n}{k}(-1)^k k\cdot x^{k-1}$$

substitute $x = 1$

$$0 = \sum_{k=0}^{n}(-1)^k k\binom{n}{k}$$

Power series

Infinite series of the form $a_0 + a_1 x + a_2 x^2 + \cdots$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

Series converges for x in the interval (-1,1)

Function contains all the information about series

Differentiate k times and substitute x=0,
we get k! times coefficient of x^k

Taylor series of the function $\frac{1}{1-x}$ at $x = 0$

Power series

$(a_0, a_1, a_2, \ldots)$: sequence of real numbers

$|a_n| \leq K^n$

For any number $x \in (-\frac{1}{K}, \frac{1}{K})$, the series

$a(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$ converges

Values of a(x) in arbitrarily small neighborhod of 0

uniquely determine $(a_0, a_1, a_2, \ldots)$

$$a_n = \frac{a^{(n)}(0)}{n!}$$

Generating functions

$(a_0, a_1, a_2, \ldots)$: sequence of real numbers

Generating function of this sequence is

the power series $a(x) = \sum_{i=0}^{\infty} a_i \cdot x^i$

Generating function basics

What is the generating function of the sequence $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots)$?

$$1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \cdots$$

$$-\frac{\ln(1-x)}{x}$$

$$x + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \cdots = -\ln(1-x)$$

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = e^x$$

Generating function toolkit: Generalized binomial theorem

$$\binom{r}{k} = \frac{r(r-1)(r-2)\ldots(r-k+1)}{k!}$$

$(1+x)^r$ is the generating function

for the sequence $\left(\binom{r}{0}, \binom{r}{1}, \binom{r}{2}, \binom{r}{3}, \ldots\right)$

The power series $\binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \ldots$

always converges for all $|x| < 1$

Negative binomial coefficients ?

$$\binom{r}{k} = (-1)^k \binom{-r+k-1}{k} = (-1)^k \binom{-r+k-1}{-r-1}$$

$$\frac{1}{(1-x)^n} = \binom{\text{n-1}}{\text{n-1}} + \binom{\text{n}}{\text{n-1}} x + \binom{\text{n+1}}{\text{n-1}} x^2 + \cdots + \binom{\text{n+k-1}}{\text{n-1}} x^k + \ldots$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

Operations on power series

- Addition

$$(a_0 + b_0, a_1 + b_1, \ldots) \text{ has generating function } a(x) + b(x)$$

- Multiplication by fixed real number

$$(\alpha a_0, \alpha a_1, \ldots) \text{ has generating function } \alpha a(x)$$

- Shifting the sequence

$$(\underbrace{0, \ldots 0}_{n\times}, a_0, a_1, \ldots) \text{ has generating function } x^n a(x)$$

- Shifting to the left

- Substituting αx for x

  $(a_0, \alpha a_1, \alpha^2 a_2 \ldots)$ has generating function $a(\alpha x)$

  $(1, 2, 4, 8, \ldots)$ has generating function ?

- Substitute x^n for x

  $(1, 1, 2, 2, 4, 4, 8, 8, \ldots)$ has generating function ?

  $$\frac{1}{1-2x^2} + \frac{x}{1-2x^2}$$

- Integration and differentiation

  $(a_0, 2a_1, 3a_2 \ldots)$ has generating function ?

  $(0, a_0, \frac{1}{2}a_1, \frac{1}{3}a_2 \ldots)$ has generating function ?

- Multiplication of generating functions