COS 341 Discrete Mathematics

Counting

Administrative Issues

- Bookstore has run out of copies of textbook.
- Readings for this week: Matousek and Nesetril, Chapter 2
  next week: Chapter 10

- Homework policy:
- All problems on a homework carry the same weight, unless stated otherwise
- All homeworks will be equally weighted
- Homeworks due in class on Wednesday
- Late homeworks submitted by 5pm Friday will be penalized 50%
- No late homeworks accepted after 5pm Friday.

How many ways are there to write a nonnegative integer m as a sum of r nonnegative integers (order is important) ?

e.g. $m = 3,\ r = 2$

$$\begin{aligned} 3 &= 0+3 \\ &= 1+2 \\ &= 2+1 \\ &= 3+0 \end{aligned}$$

How many ordered r-tuples $(i_1, i_2, \ldots, i_r)$ of nonnegative integers satisfy the equation

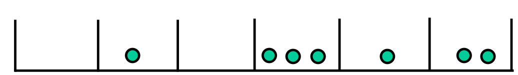
$$i_1 + i_2 + \cdots + i_r = m$$

$$i_1 + i_2 + \cdots + i_r = m$$

m indistinguishable balls and r boxes

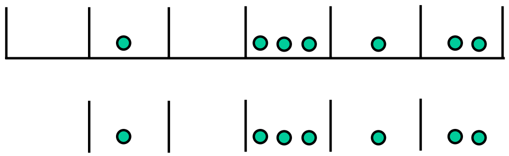
How many ways of placing m balls in r boxes ?

Each placement gives a solution of the equation

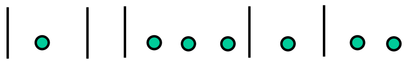


$0 \;+\; 1 \;+\; 0 \;+\; 3 \;+\; 1 \;+\; 2 \;=\; 7$

Placing balls in boxes



m balls r-1 walls



Every arrangement of m balls and r-1 walls gives a unique placement of m balls into r boxes

m+r-1 objects arranged in a row,

m balls, r-1 walls

Number of choices of r-1 positions amongst m+r-1

$$\binom{m+r-1}{r-1}$$

Number of solutions of the equation

$$i_1 + i_2 + \cdots + i_r = m$$

Properties of Binomial coefficients

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

A messy proof

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(n-k)!(k-1)!} + \frac{(n-1)!}{(n-k-1)!k!} \\
&= \frac{(n-1)!}{(n-k-1)!(k-1)!}\left(\frac{1}{n-k} + \frac{1}{k}\right) \\
&= \frac{(n-1)!}{(n-k-1)!(k-1)!}\left(\frac{n}{(n-k)k}\right) \\
&= \frac{n!}{(n-k)!k!} = \binom{n}{k}
\end{aligned}$$

A more elegant proof

Fix an element a of an n element set

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

Number of k-element subsets that include a

Number of k-element subsets that do not include a

Total number of k-element subsets

Pascal's triangle

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$$\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$$

What is the sum of numbers in the nth row of Pascal's triangle ?

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Number of k-element subsets of n-element set

Total number of subsets of n-element set

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$= \binom{n}{0} y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \ldots + \binom{n}{n-1} x^{n-1} y + \binom{n}{n} x^n$$

Proof of Binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot\ldots\cdot(x+y)}_{n\text{ times}}$$

Coefficient of $x^k y^{n-k}$

$=$ number of ways of picking k x's from n terms

$= \binom{n}{k}$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n$$

Proofs using the binomial theorem

Substitute $x = 1$ in

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Proofs using the binomial theorem

Substitute $x = -1$ in

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

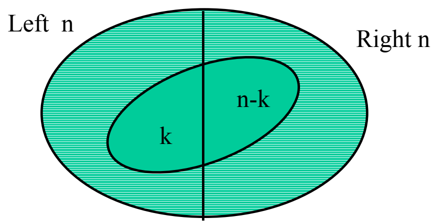
$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots$$

$$2^n = 2\left[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots\right]$$

Further identities

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$$

$$\sum_{k=0}^{n}\binom{n}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$$



How many distinct words can you obtain by permuting the letters of M I S S I S S I P P I ?

11 letters: 1 M, 4 I, 4 S, 2 P

$M_1\, I_1\, S_1\, S_2\, I_2\, S_3\, S_4\, I_3\, P_1\, P_2\, I_4$

11! permutations

How many indexed words give you a particular unindexed word ?

e.g. S I P I S M S I P I S

4! ways to place indices of I

4! ways to place indices of S

2! ways to place indices of P

1! way for indices of M

number of distinct words $=\dfrac{11!}{4!4!2!1!}$

Generalization

n objects of m different kinds

k_i indistinguishable objects of ith kind

$$k_1+k_2+\ldots+k_m=n$$

Then the total number of distinct arrangements is

$$\frac{n!}{k_1!k_2!\ldots k_m!} \qquad \binom{n}{k_1,k_2,\ldots,k_m}$$

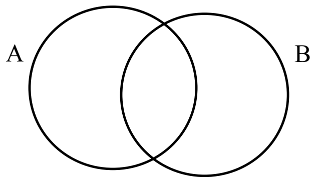
Multinomial coefficient

$$m=2 \qquad \binom{n}{k,n-k}=\binom{n}{k}$$

Multinomial theorem

$$(x_1+x_2+\cdots+x_m)^n=\sum_{\substack{k_1+\ldots+k_m=n\\ k_1,\ldots,k_m\geq 0}}\binom{n}{k_1,k_2,\ldots,k_m}x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m}$$

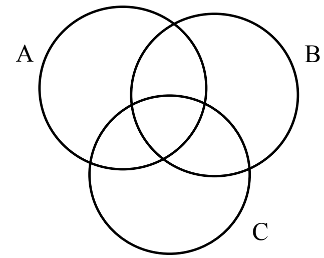
Inclusion-Exclusion principle



Given |A|, |B|, $|A \cap B|$, what is $|A \cup B|$?

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Inclusion-Exclusion principle



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Inclusion-Exclusion principle

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i_1 \le i_2 \le n}^{n} |A_{i_1} \cap A_{i_2}| + \sum_{1 \le i_1 \le i_2 \le i_3 \le n}^{n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Proof of inclusion-exclusion principle

Consider an element $x \in A_1 \cup A_2 \cup \cdots \cup A_n$

Contribution to LHS = 1

What is the contribution to RHS ?

Suppose x belongs to j sets.

Rename sets to be $A_1, A_2, \ldots, A_j$

x appears in intersection of every

k-tuple of sets amongst $A_1, A_2, \ldots, A_j$

Proof of inclusion-exclusion principle

x appears in intersection of every

k-tuple of sets amongst $A_1, A_2, \ldots, A_j$

contribution of x to RHS

$$= \sum_{i=1}^{k} (-1)^{k-1} \binom{j}{k}$$

$$= j - \binom{j}{2} + \binom{j}{3} - \cdots + (-1)^{j-1} \binom{j}{j}$$

$$= 1$$