Administrative Issues

- Bookstore has run out of copies of textbook.
- Readings are now on web page.
- Readings for this week: Matousek and Nesetril, 2.1 – 2.6

Rolling dice

- Suppose I roll a white die and a black die
- Number of possible outcomes = $6 \times 6 = 36$
- How many outcomes where dice show different values?

Different outcomes

- $S$ = Set of all outcomes where the dice show different values
- $A_i$ = set of outcomes where the black die says $i$ and the white die says something else.

$$|S| = \bigcup_{i=1}^{6} A_i = \sum_{i=1}^{6} |A_i| = \sum_{i=1}^{6} 5 = 30$$
Different outcomes: take 2

- $S \equiv$ Set of all outcomes where the dice show different values
- $T \equiv$ set of outcomes where dice agree.

$$|S \cup T| = \text{# of outcomes} = 36$$
$$|S| + |T| = 36 \quad |T| = 6$$
$$|S| = 36 - 6 = 30$$

Another approach

- $S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die.
- $L \equiv$ set of all outcomes where the black die shows a larger number than the white die.

$$|S| + |L| = 30$$
- It is clear by symmetry that $|S| = |L|$.
- Therefore $|S| = 15$

• How many outcomes where the black die says a smaller number than the white die?

- $A_i \equiv$ set of outcomes where the black die says $i$ and the white die says something larger.

$$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$
$$|S| = 5 + 4 + 3 + 2 + 1 + 0 = 15$$

• What do we mean by symmetry?

- To formalize this idea, we show a correspondence from $S$ to $L$.

- We put each outcome in $S$ in correspondence with an outcome in $L$ by swapping the color of the dice.

black 2, white 5 is mapped to black 5, white 2
- Each outcome in $S$ matched with exactly one outcome in $L$, with none left over.
- Thus $|S| = |L|$
• Let $f:A \to B$ be a function from a set $A$ to a set $B$.

• $f$ is 1-1 if and only if
  \[ \forall x, y \in A, \ x \neq y \Rightarrow f(x) \neq f(y) \]

• $f$ is onto if and only if
  \[ \forall z \in B \ \exists x \in A \ f(x) = z \]

\[ \exists 1-1 f:A \to B \Rightarrow |A| \leq |B| \]

\[ \exists \text{onto } f:A \to B \Rightarrow |A| \geq |B| \]
∃ 1-1 onto \( f : A \rightarrow B \Rightarrow |A| = |B| \)

If you have a bijection between two sets, then they have the same size.

Sets and Subsets

- \( X = \{a, b, c\} \)
- Subsets of \( X \) are \( \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\} \)
  \( \{a, b, c\}, \emptyset \)
- How many subsets does \( X \) have?

Counting subsets

Theorem: Any \( n \)-element set \( X \) has exactly \( 2^n \) subsets

Proof: (by induction on size of \( X \))
- For \( X = \emptyset \), \( \emptyset \) is the only subset
  The statement holds for \( n=0 \)
- Suppose the statement holds for \( n=k \)
- Consider a set \( X \) of size \( k+1 \)
  Fix an element \( a \in X \)
  Divide the subsets of \( X \) into two classes

Two classes

Consider a set \( X \) of size \( k+1 \)
Fix an element \( a \in X \)
Divide the subsets of \( X \) into two classes
Second class has $2^k$ subsets.
Bijection from first class to second class
Map subset $S$ to $S\setminus \{a\}$

Finishing up ….

- Both classes have the same size.
- Second class has size $2^k$
- Total number of subsets of $X = 2^k + 2^k = 2^{k+1}$
- Statement true by induction.

An alternate proof
Map subsets to bit sequences
$X=\{a,b,c,d,e\}$, subset $\{b,d,e\}$

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

In general, for set $X=\{x_1, x_2, x_3, \ldots, x_n\}$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$\ldots$</th>
<th>$x_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_1$</td>
<td>$b_2$</td>
<td>$b_3$</td>
<td>$\ldots$</td>
<td>$b_n$</td>
</tr>
</tbody>
</table>

Subsets and bit sequences

- **Bijection** between subsets of an $n$ element set and bit sequences of length $n$
- $f$ is 1-1
- $f$ is onto

- Number of bit sequences of length $n$ is $2^n$
- Number of subsets of an $n$ element set is $2^n$
Odd and even

- Consider an \( n \) element set \( X \)
- How many subsets of odd size does \( X \) have?
- Fix an element \( a \in X \)
- Bijection between odd subsets of \( X \) and subsets of \( X \setminus \{a\} \)

Hence number of odd subsets is exactly \( 2^{n-1} \)

Why is this correct?

Can two different odd subsets map to the same subset of \( X \setminus \{a\} \)?

Counting with choices

- We want to count the number of objects in a set
- We give a process to select an object in the set by making a sequence of choices
- Each distinct sequence of choices leads to an object in the set
- Each object in the set can be produced by exactly one sequence of choices
- Suppose there are \( P_1 \) possibilities for 1st choice, \( P_2 \) possibilities for 2nd choice, …., \( P_n \) possibilities for \( n^{th} \) choice
- Then there are \( P_1 \times P_2 \times \ldots \times P_n \) objects in the set
Ordering a deck of cards

- How many different orderings of a deck of cards?
- 52 choices for the 1st card
- 51 choices for the 2nd card
- 50 choices for the 3rd card
- ...
- 1 choice for the 52nd card
- Number of orderings = $52 \times 51 \times 50 \times \ldots \times 1$
  
  = 52!
  
  (52 factorial)

Permutations

- A permutation or an arrangement of $n$ objects is an ordering of the objects.

- The number of permutations of $n$ distinct objects is $n!$

Permutations of $r$ out of $n$

- The number of ways of ordering, permuting, or arranging $r$ out of $n$ objects.
- $n$ choices for first place, $n-1$ choices for second place, ...
- $n \times (n-1) \times (n-2) \times \ldots \times (n-(r-1))$

\[
= \frac{n!}{(n-r)!}
\]

- How many sequences of 5 letters contain at least two of the same letter?
Counting the opposite

- How many sequences of 5 letters contain at least two of the same letter?

\[= \text{no. 5-letter sequences} - \text{no. 5-letter sequences with all letters distinct}\]

\[= 26^5 - 26 \times 25 \times 24 \times 23 \times 22\]

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

\[-52 \times 51\]

How many unordered pairs?

- \(52 \times 51 / 2 \leftarrow \text{divide by overcount}\)
  
  Each unordered pair is listed twice on a list of the ordered pairs, but we consider the ordered pairs to be the same.

Ordered Versus Unordered

- From a deck of 52 cards how many ordered 5 card sequences can be formed?
  
  \[-52 \times 51 \times 50 \times 49 \times 48\]

- How many orderings of 5 cards?
  
  \[-5!\]

- How many unordered 5 card hands?

\[52 \times 51 \times 50 \times 49 \times 48 / 5! = 2,598,960\]

A combination or choice of r out of n objects is an (unordered) set of r of the n objects.

- The number of r combinations of n objects:

\[
\frac{n!}{r!(n-r)!} = \binom{n}{r}
\]

\(n \text{ choose } r\)