COS 341 Discrete Mathematics

Counting

Administrative Issues

- Bookstore has run out of copies of textbook.
- Readings are now on web page.
- Readings for this week: Matousek and Nesetril, 2.1 – 2.6

Rolling dice

- Suppose I roll a white die and a black die
- Number of possible outcomes = $6 \times 6 = 36$
- How many outcomes where dice show different values ?

Different outcomes

- $S \equiv$ Set of all outcomes where the dice show different values

- $A_i \equiv$ set of outcomes where the black die says i and the white die says something else.

$$|S| = \left|\bigcup_{i=1}^{6} A_i\right| = \sum_{i=1}^{6} |A_i| = \sum_{i=1}^{6} 5 = 30$$

Different outcomes: take 2

- S ≡ Set of all outcomes where the dice show different values

- T ≡ set of outcomes where dice agree.

$$|S \cup T| = \text{\#of outcomes} = 36$$

$$|S| + |T| = 36 \qquad |T| = 6$$

$$|S| = 36 - 6 = 30$$

- How many outcomes where the black die says a smaller number than the white die ?

- A_i ≡ set of outcomes where the black die says i and the white die says something larger.

$$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$

$$|S| = 5 + 4 + 3 + 2 + 1 + 0 = 15$$

Another approach

- $S \equiv$ Set of all outcomes where the black die shows a smaller number than the white die.

- $L \equiv$ set of all outcomes where the black die shows a larger number than the white die.

$$|S| + |L| = 30$$

- It is clear by symmetry that $|S| = |L|$.

- Therefore $|S| = 15$

- What do we mean by symmetry ?

- To formalize this idea, we show a correspondence from S to L.

- We put each outcome in S in correspondence with an outcome in L by swapping the color of the dice.

  black 2, white 5 is mapped to black 5, white 2

- Each outcome in S matched with exactly one outcome in L, with none left over.
- Thus $|S| = |L|$

- Let $f:A \to B$ be a function from a set A to a set B.

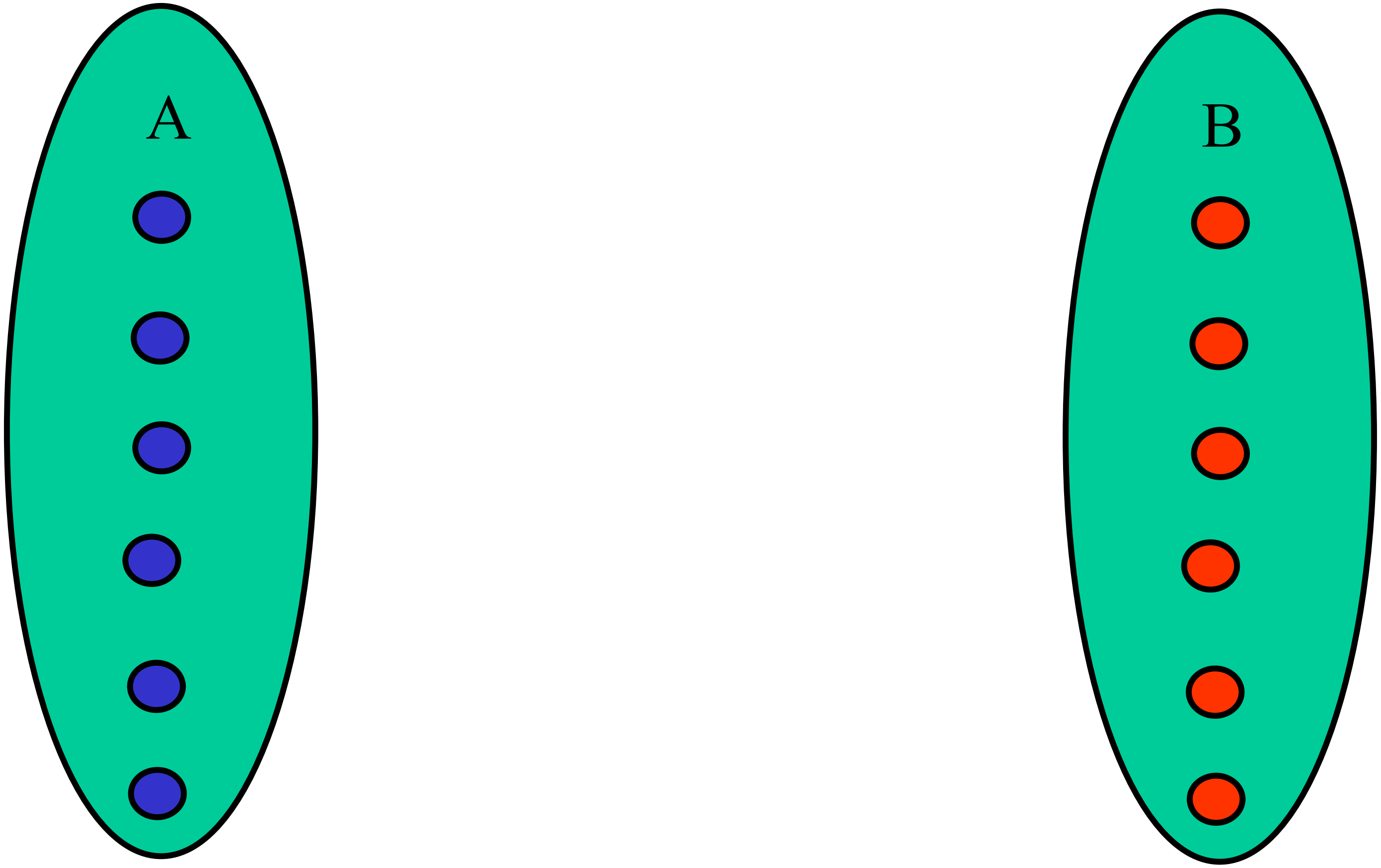
- f is 1-1 if and only if

  $$\forall x,y \in A,\ \ x \neq y \Rightarrow f(x) \neq f(y)$$

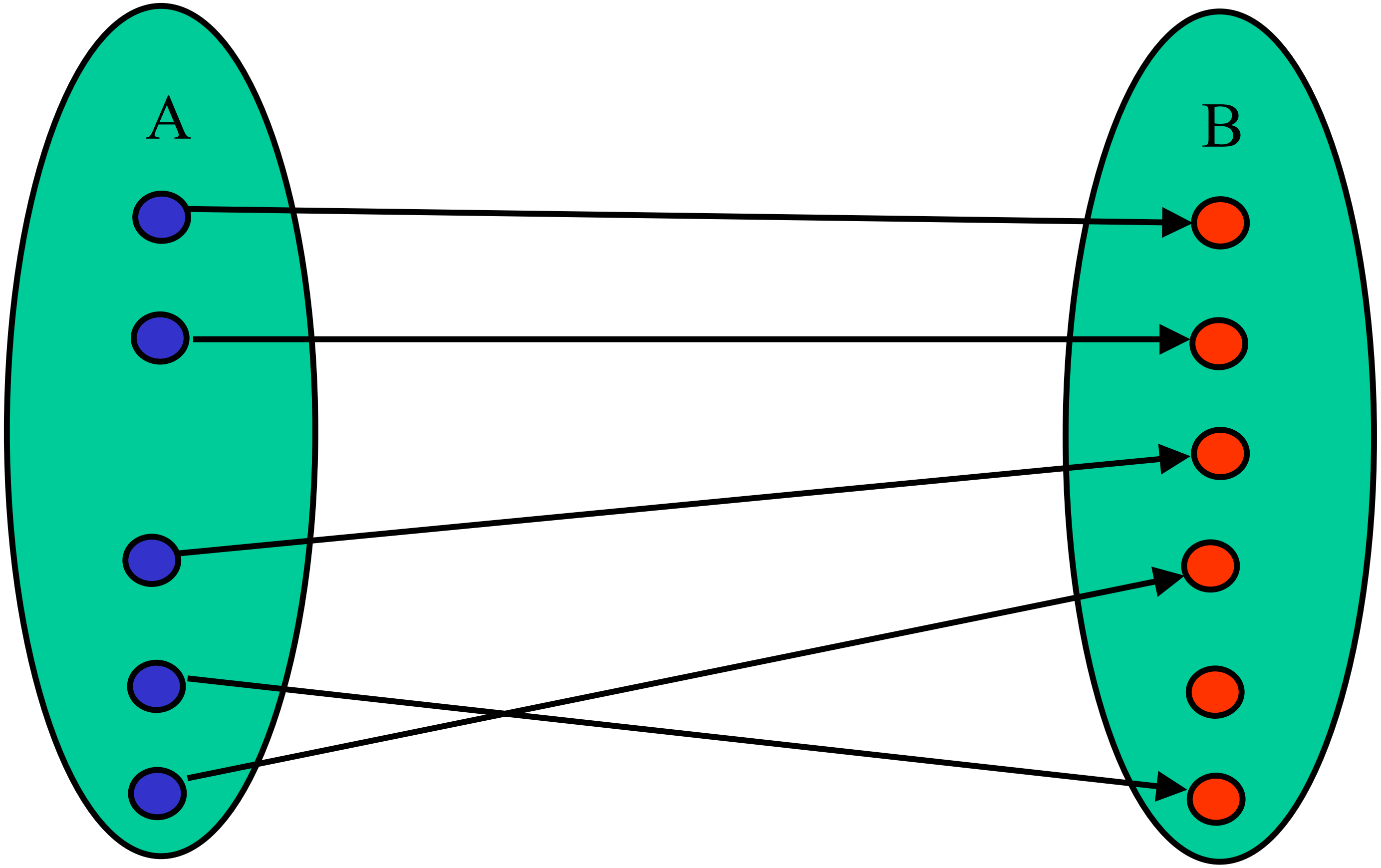
- f is onto if and only if

  $$\forall z \in B\ \ \exists x \in A\ \ f(x) = z$$

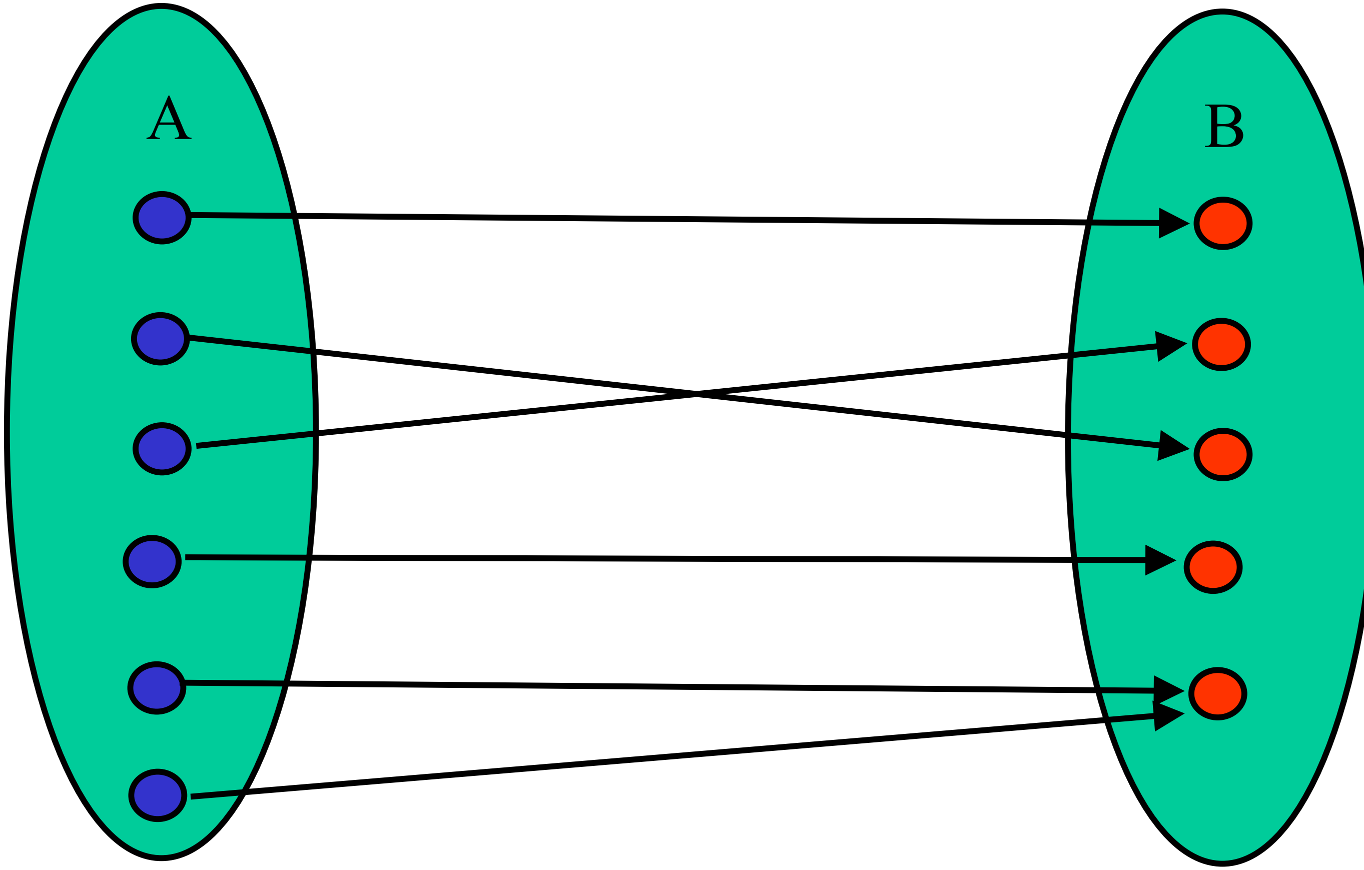
Mappings between finite sets



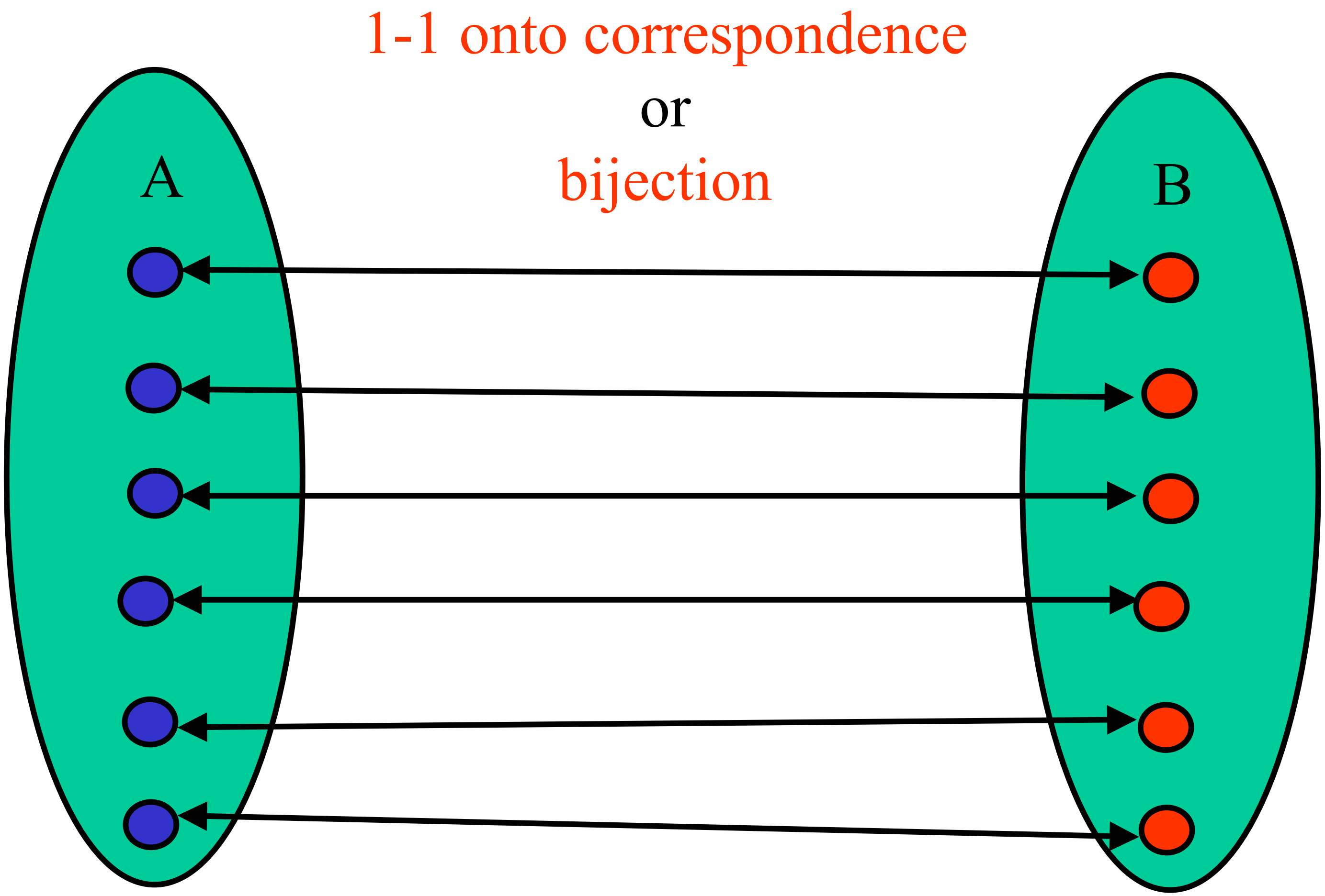
$\exists$ 1-1 $f{:}A \to B \Rightarrow |A| \leq |B|$



$\exists$ onto $f:A \to B \Rightarrow |A| \geq |B|$



$\exists$ 1-1 onto $f: A \rightarrow B \Rightarrow |A| = |B|$



If you have a bijection between two sets, then they have the same size

Sets and Subsets

- X = {a,b,c}
- Subsets of X are {a}, {b}, {c}, {a,b}, {a,c}, {b,c}
  {a,b,c}, $\varnothing$

- How many subsets does X have ?

Counting subsets

Theorem: Any n-element set X has exactly 2^n subsets

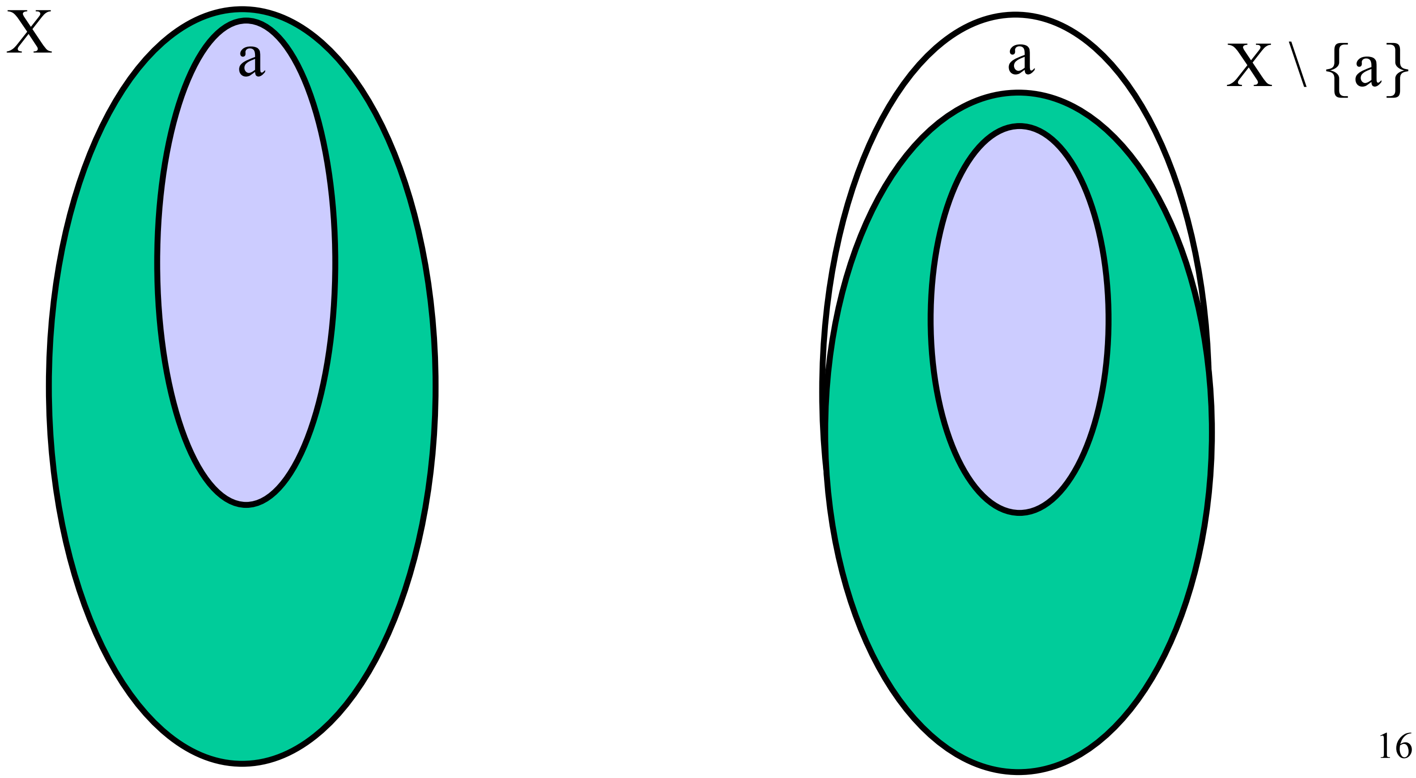
Proof: (by induction on size of X)

- For $X = \emptyset$, $\emptyset$ is the only subset
  The statement holds for $n=0$

- Suppose the statement holds for $n=k$
- Consider a set X of size $k+1$
  Fix an element $a \in X$
  Divide the subsets of X into two classes

Two classes

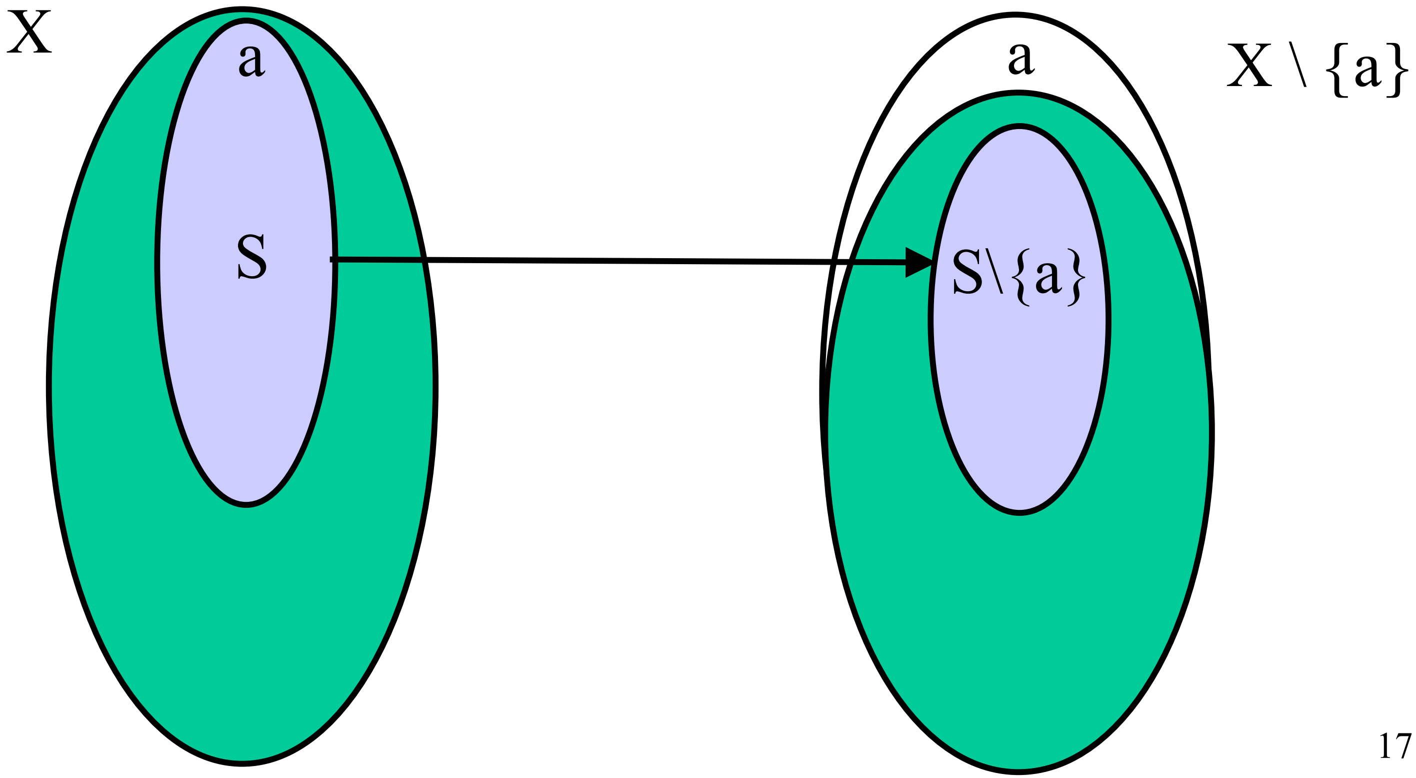
Consider a set X of size k+1
Fix an element $a \in X$
Divide the subsets of X into two classes



Second class has 2^k subsets.

Bijection from first class to second class

Map subset S to S\{a}



Finishing up ….

- Both classes have the same size.
- Second class has size 2^k
- Total number of subsets of X = $2^k + 2^k = 2^{k+1}$
- Statement true by induction.

An alternate proof

Map subsets to bit sequences

$X=\{a,b,c,d,e\}$, subset $\{b,d,e\}$

a	b	c	d	e
0	1	0	1	1

In general, for set $X=\{x_1\ x_2\ x_3 \ldots x_n\}$

x_1	x_2	x_3	….	x_n
b_1	b_2	b_3	….	b_n

$f(S) = b$

Subsets and bit sequences

x_1	x_2	x_3	….	x_n
b_1	b_2	b_3	….	b_n

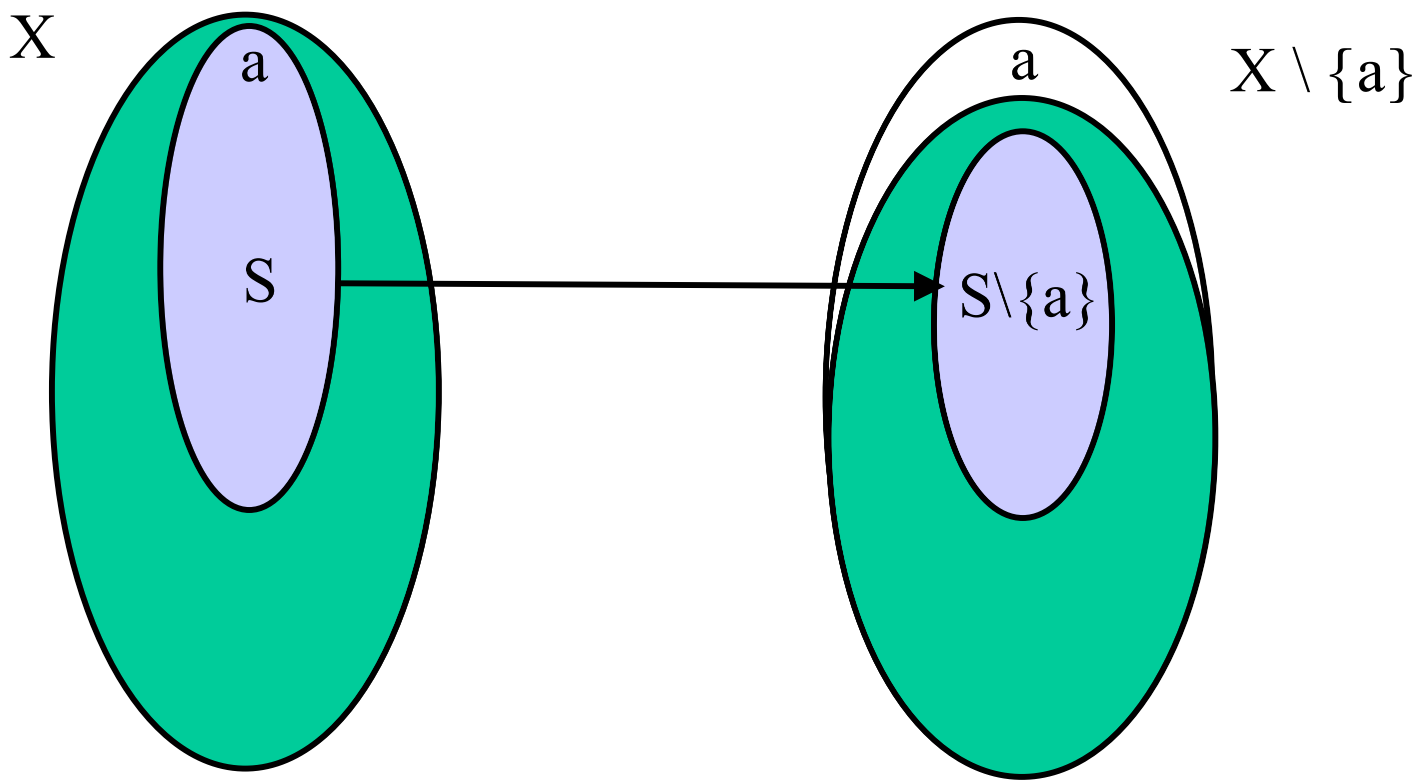
$f(S) = b$

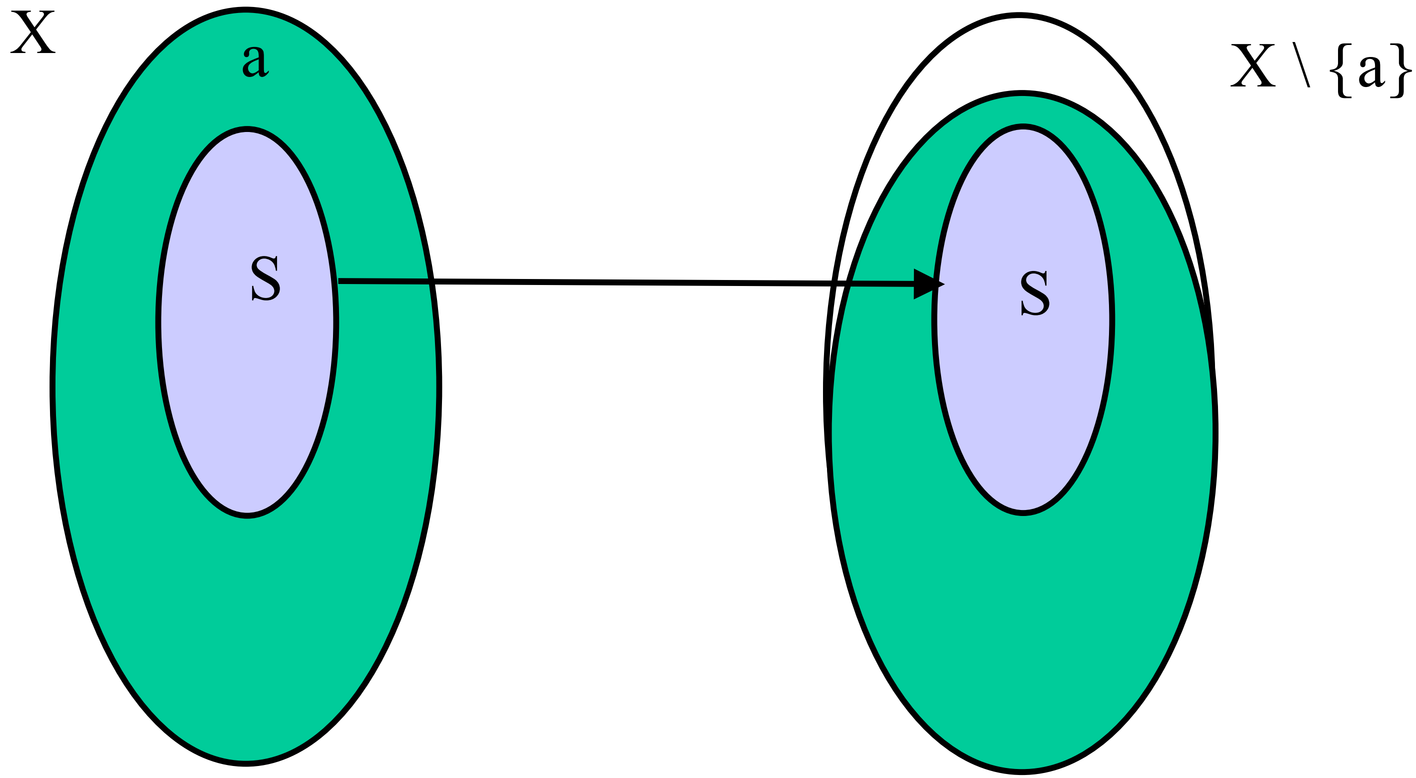
- Bijection between subsets of an n element set and bit sequences of length n
- f is 1-1
- f is onto

- Number of bit sequences of length n is 2^n
- Number of subsets of an n element set is 2^n

Odd and even

- Consider an n element set X
- How many subsets of odd size does X have ?
- Fix an element $a \in X$
- Bijection between odd subsets of X and subsets of $X \setminus \{a\}$

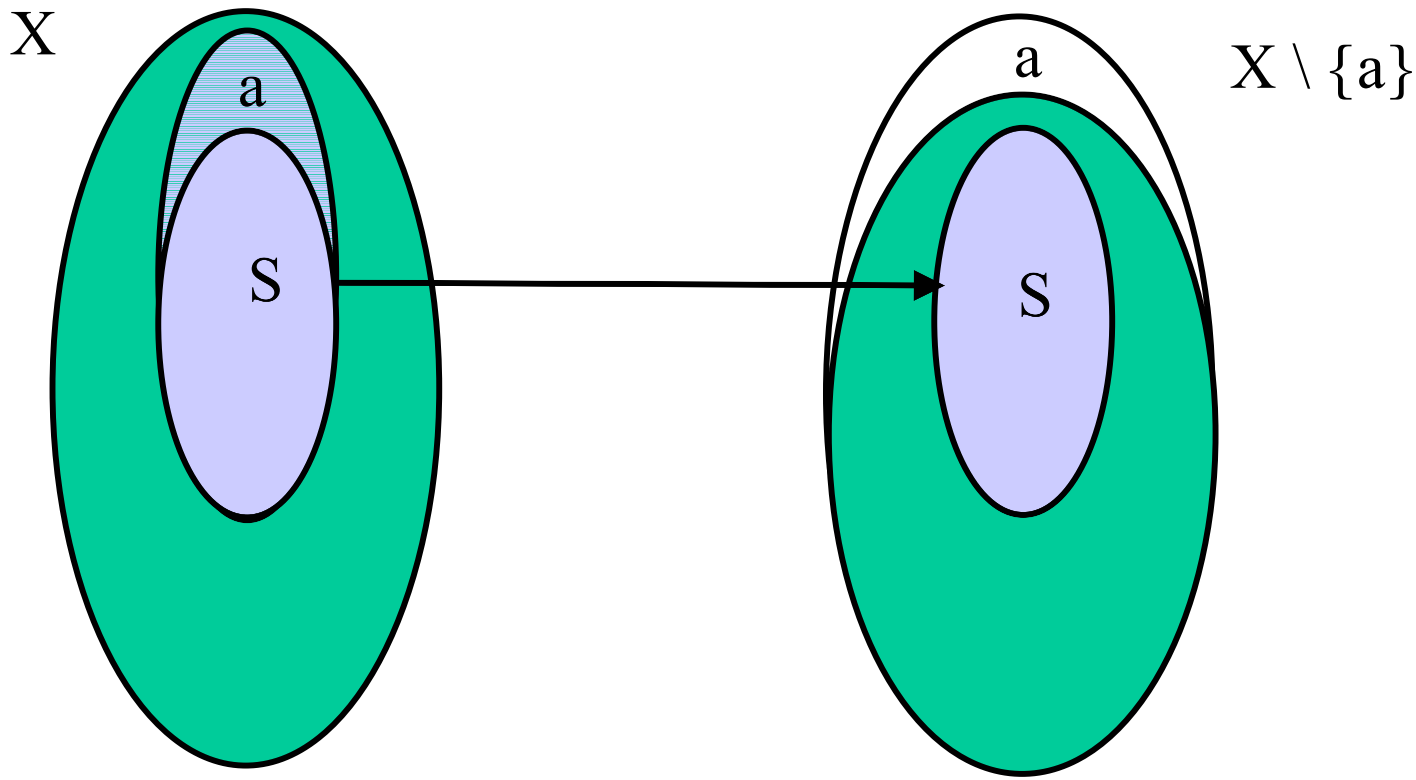




Hence number of odd subsets is exactly 2^{n-1}

Why is this correct ?

Can two different odd subsets map to the same subset of X \ {a} ?



Counting with choices

- We want to count the number of objects in a set
- We give a process to select an object in the set by making a sequence of choices
  - Each distinct sequence of choices leads to an object in the set
  - Each object in the set can be produced by exactly one sequence of choices
- Suppose there are P_1 possibilities for 1st choice,
  P_2 possibilities for 2^{nd} choice,
  …. P_n possibilities for n^{th} choice
- Then there are $P_1 \times P_2 \times \ldots \times P_n$ objects in the set

Ordering a deck of cards

- How many different orderings of a deck of cards ?
- 52 choices for the 1^{st} card
- 51 choices for the 2^{nd} card
- 50 choices for the 3^{rd} card
- …
- 1 choice for the 52^{nd} card
- Number of orderings = $52 \times 51 \times 50 \times \ldots \times 1$

  $= 52\,!$

  (52 factorial)

Permutations

- A permutation or an arrangement of n objects is an ordering of the objects.

- The number of permutations of n distinct objects is n!

Permutations of r out of n

- The number of ways of ordering, permuting, or arranging r out of n objects.
- n choices for first place, n-1 choices for second place, . . .
- $n \times (n-1) \times (n-2) \times \ldots \times (n-(r-1))$

$$= \frac{n!}{(n-r)!}$$

- How many sequences of 5 letters contain at least two of the same letter ?

Counting the opposite

- How many sequences of 5 letters contain at least two of the same letter ?

= no. 5-letter sequences

– no. 5-letter sequences with all letters distinct

$= 26^5 - 26 \times 25 \times 24 \times 23 \times 22$

Ordered Versus Unordered

From a deck of 52 cards how many ordered pairs can be formed?

– 52×51

How many unordered pairs?

• $52 \times 51 / 2$ ← divide by overcount

Each unordered pair is listed twice on a list of the ordered pairs, but we consider the ordered pairs to be the same.

Ordered Versus Unordered

- From a deck of 52 cards how many ordered 5 card sequences can be formed?
  - $52 \times 51 \times 50 \times 49 \times 48$
- How many orderings of 5 cards?
  - $5!$
- How many unordered 5 card hands?

$$52 \times 51 \times 50 \times 49 \times 48 / 5! = 2{,}598{,}960$$

A <u>combination</u> or <u>choice</u> of r out of n objects is an (unordered) set of r of the n objects.

- The number of r combinations of n objects:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

n choose r