

COS 341 Discrete Mathematics

Counting

Administrative Issues

- Bookstore has run out of copies of textbook.
- Readings are now on web page.
- Readings for this week: Matousek and Neseřil, 2.1 – 2.6

Rolling dice

- Suppose I roll a white die and a black die
- Number of possible outcomes = $6 \times 6 = 36$
- How many outcomes where dice show different values ?

Different outcomes

- $S \equiv$ Set of all outcomes where the dice show different values
- $A_i \equiv$ set of outcomes where the black die says i and the white die says something else.

$$|S| = \left| \bigcup_{i=1}^6 A_i \right| = \sum_{i=1}^6 |A_i| = \sum_{i=1}^6 5 = 30$$

Different outcomes: take 2

- $S \equiv$ Set of all outcomes where the dice show different values
- $T \equiv$ set of outcomes where dice agree.

$$|S \cup T| = \text{\#of outcomes} = 36$$

$$|S| + |T| = 36 \quad |T| = 6$$

$$|S| = 36 - 6 = 30$$

- How many outcomes where the black die says a **smaller** number than the white die ?
- $A_i \equiv$ set of outcomes where the black die says i and the white die says something **larger**.

$$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$

$$|S| = 5 + 4 + 3 + 2 + 1 + 0 = 15$$

Another approach

- $S \equiv$ Set of all outcomes where the black die shows a **smaller** number than the white die.
- $L \equiv$ set of all outcomes where the black die shows a **larger** number than the white die.

$$|S| + |L| = 30$$

- It is clear by **symmetry** that $|S| = |L|$.
- Therefore $|S| = 15$

- What do we mean by **symmetry** ?
- To formalize this idea, we show a correspondence from **S** to **L**.
- We put each outcome in **S** in correspondence with an outcome in **L** by **swapping** the color of the dice.

black 2, white 5 is mapped to black 5, white 2

- Each outcome in **S** matched with exactly one outcome in **L**, with none left over.
- Thus $|S| = |L|$

- Let $f:A \rightarrow B$ be a function from a set A to a set B .

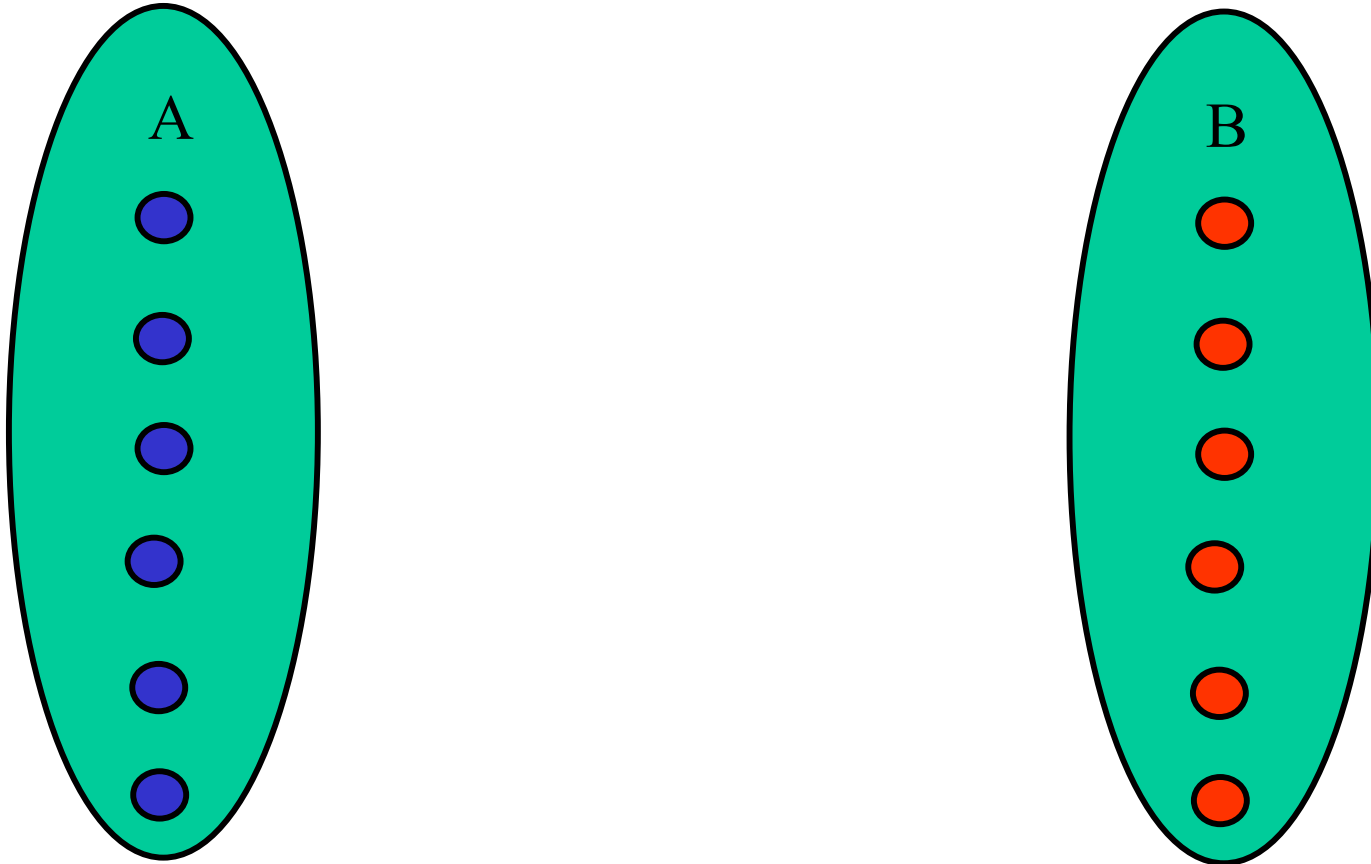
- f is **1-1** if and only if

$$\forall x, y \in A, x \neq y \Rightarrow f(x) \neq f(y)$$

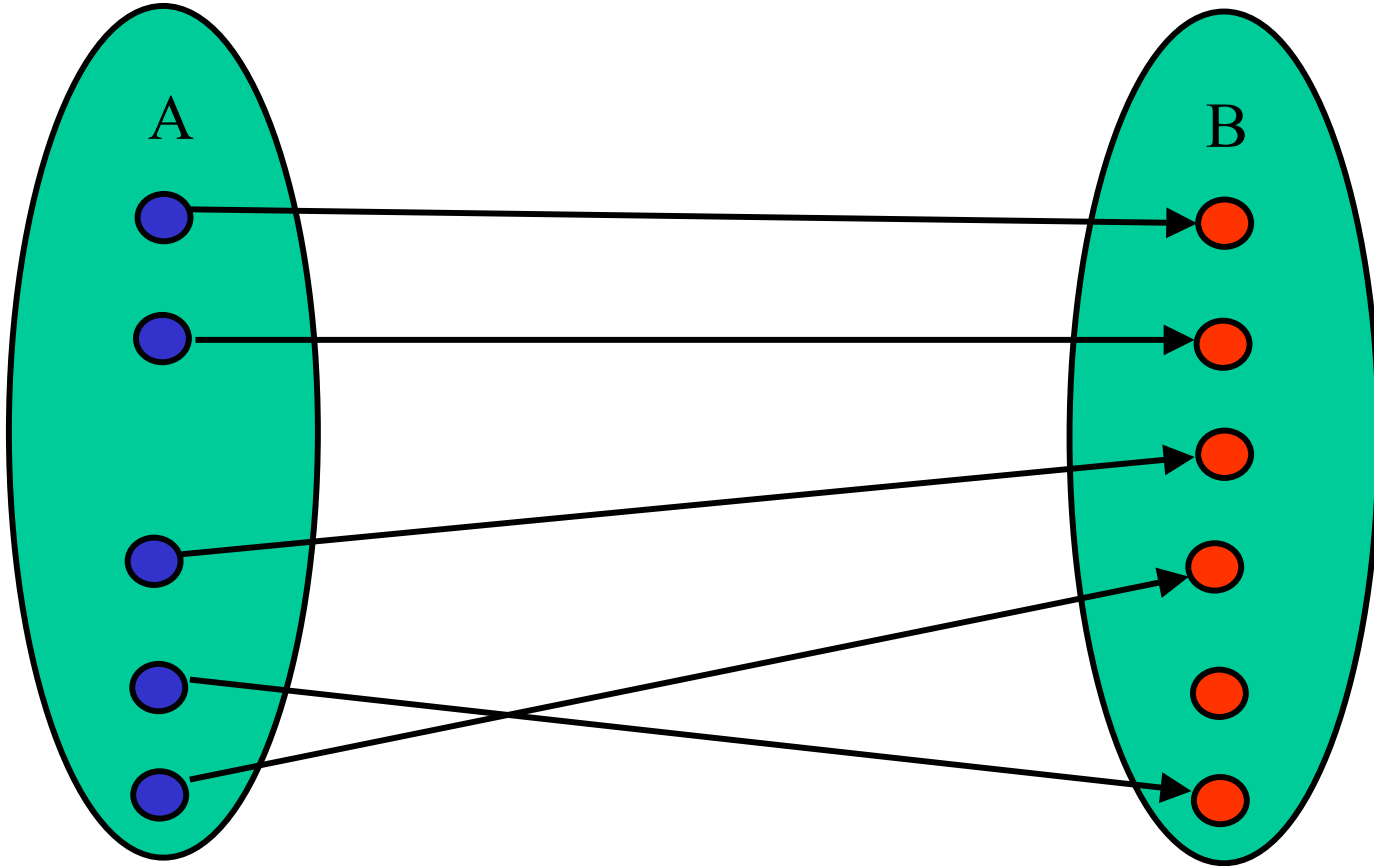
- f is **onto** if and only if

$$\forall z \in B \exists x \in A f(x) = z$$

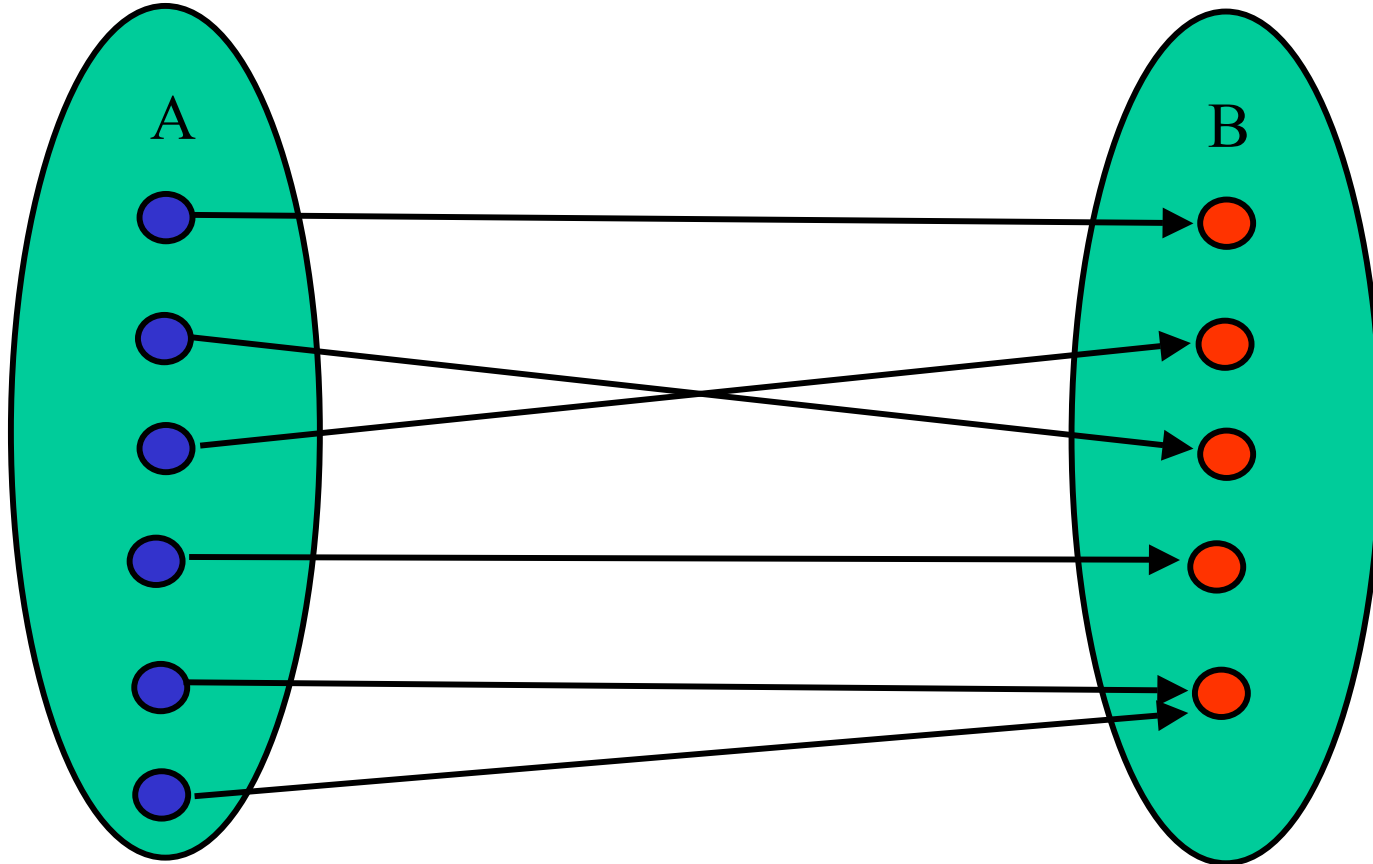
Mappings between finite sets



$$\exists 1-1 f:A \rightarrow B \Rightarrow |A| \leq |B|$$



$$\exists \text{ onto } f:A \rightarrow B \Rightarrow |A| \geq |B|$$

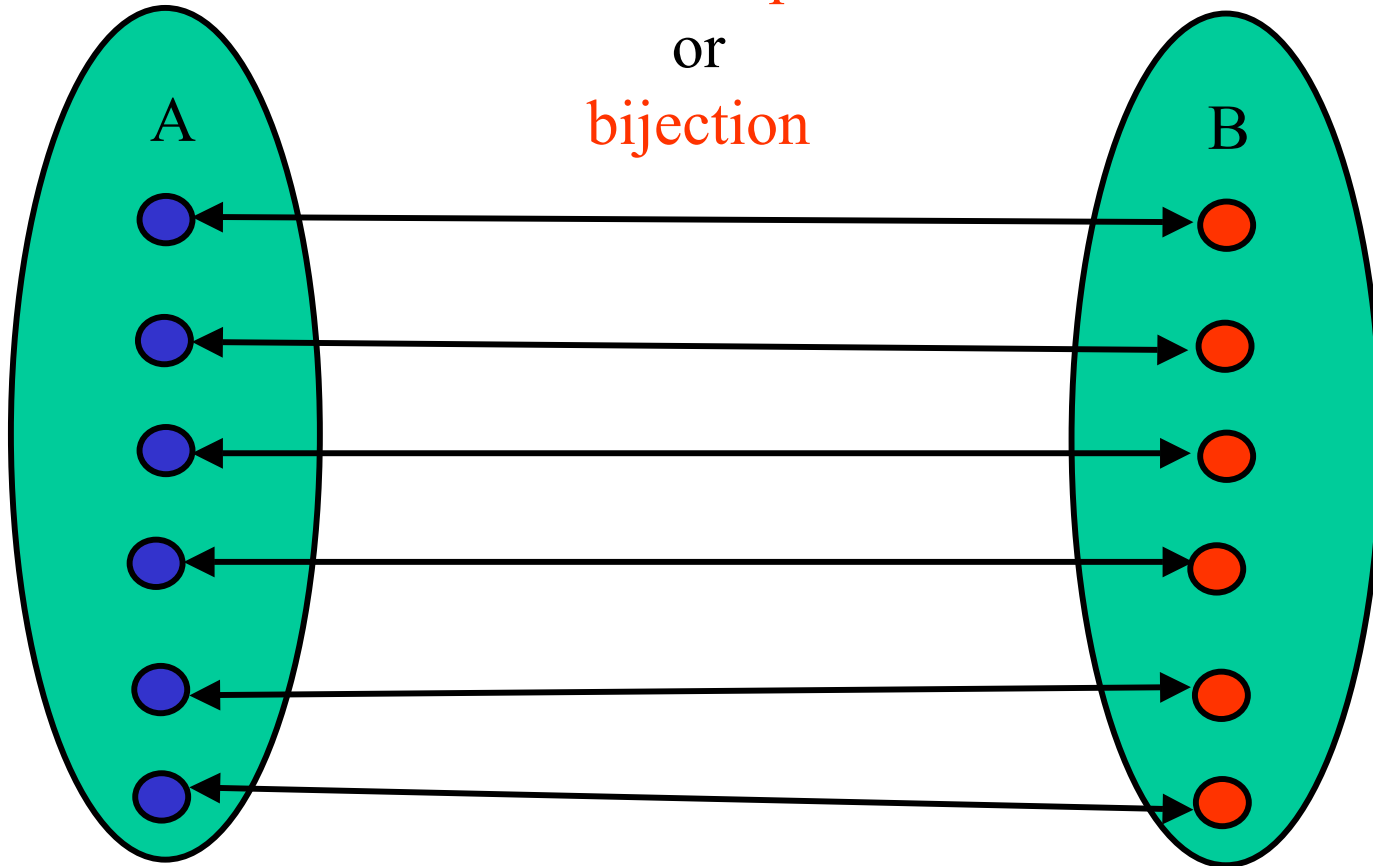


\exists 1-1 onto $f:A \rightarrow B \Rightarrow |A| = |B|$

1-1 onto correspondence

or

bijection



If you have a **bijection** between two sets,
then they have the **same size**

Sets and Subsets

- $X = \{a,b,c\}$
- Subsets of X are $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$
 $\{a,b,c\}$, \emptyset
- How many subsets does X have ?

Counting subsets

Theorem: Any n -element set X has exactly 2^n subsets

Proof: (by induction on size of X)

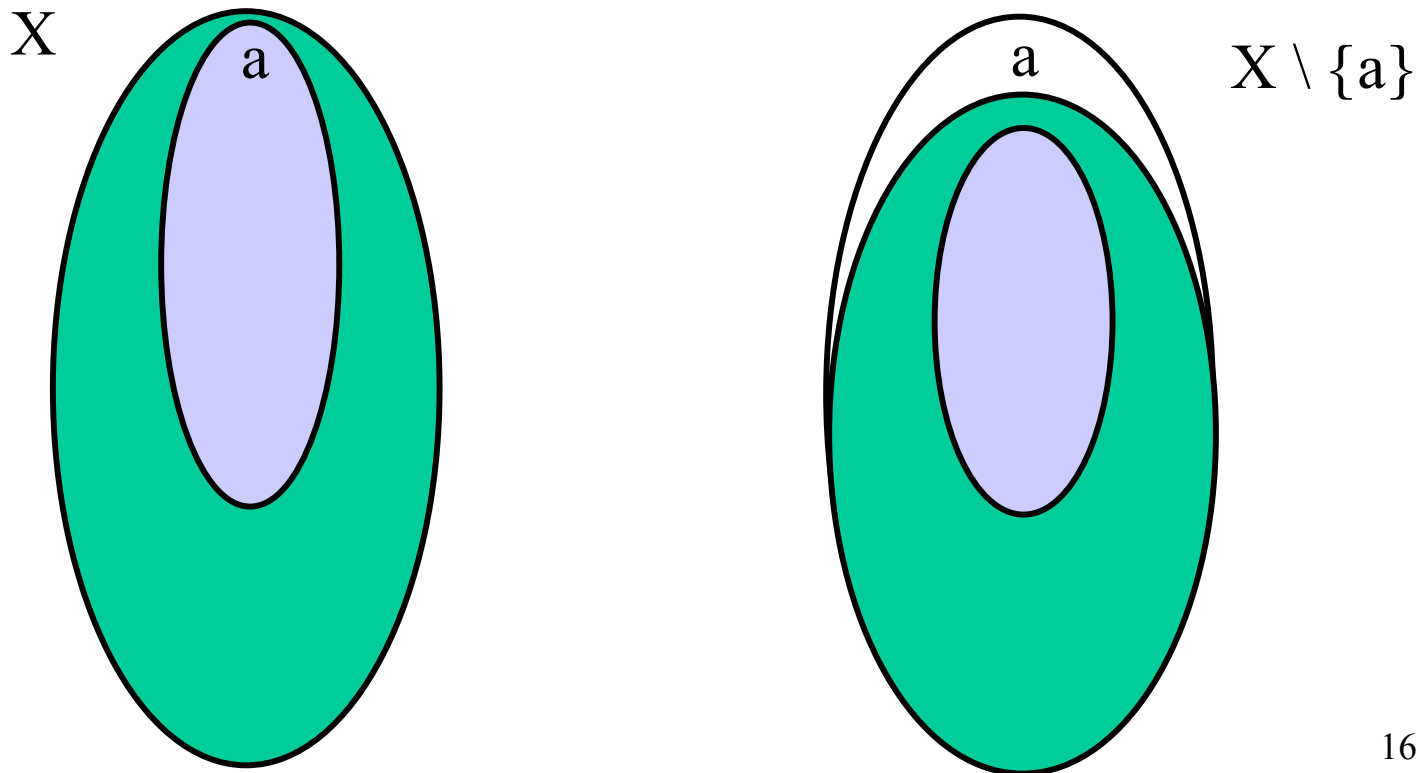
- For $X = \emptyset$, \emptyset is the only subset
The statement holds for $n=0$
- Suppose the statement holds for $n=k$
- Consider a set X of size $k+1$
Fix an element $a \in X$
Divide the subsets of X into two classes

Two classes

Consider a set X of size $k+1$

Fix an element $a \in X$

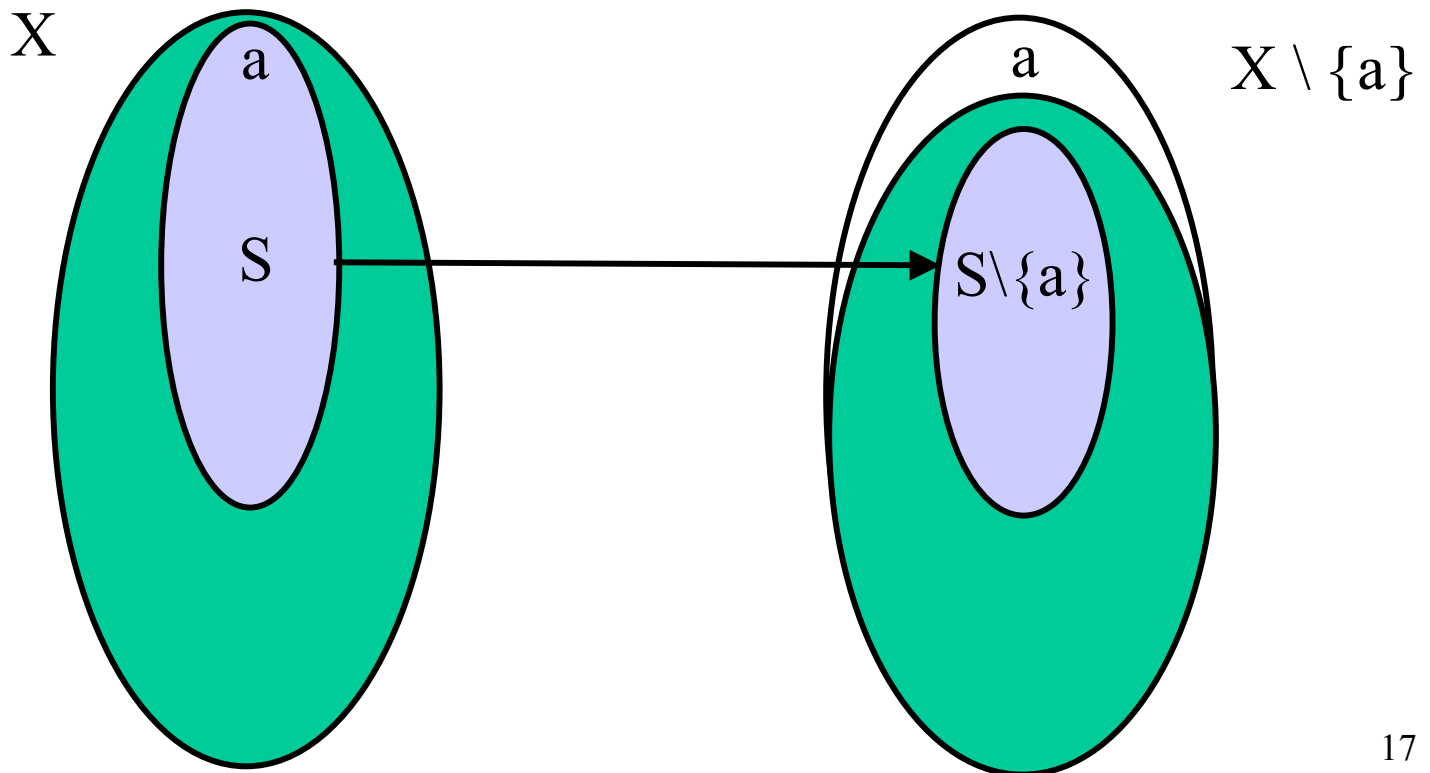
Divide the subsets of X into two classes



Second class has 2^k subsets.

Bijection from first class to second class

Map subset S to $S \setminus \{a\}$



Finishing up

- Both classes have the same size.
- Second class has size 2^k
- Total number of subsets of $X = 2^k + 2^k = 2^{k+1}$
- Statement true by induction.

An alternate proof

Map subsets to bit sequences

$X = \{a, b, c, d, e\}$, subset $\{b, d, e\}$

a	b	c	d	e
0	1	0	1	1

In general, for set $X = \{x_1 x_2 x_3 \dots x_n\}$

x_1	x_2	x_3	x_n
b_1	b_2	b_3	\dots	b_n

$$f(S) = b$$

Subsets and bit sequences

x_1	x_2	x_3	x_n
b_1	b_2	b_3	b_n

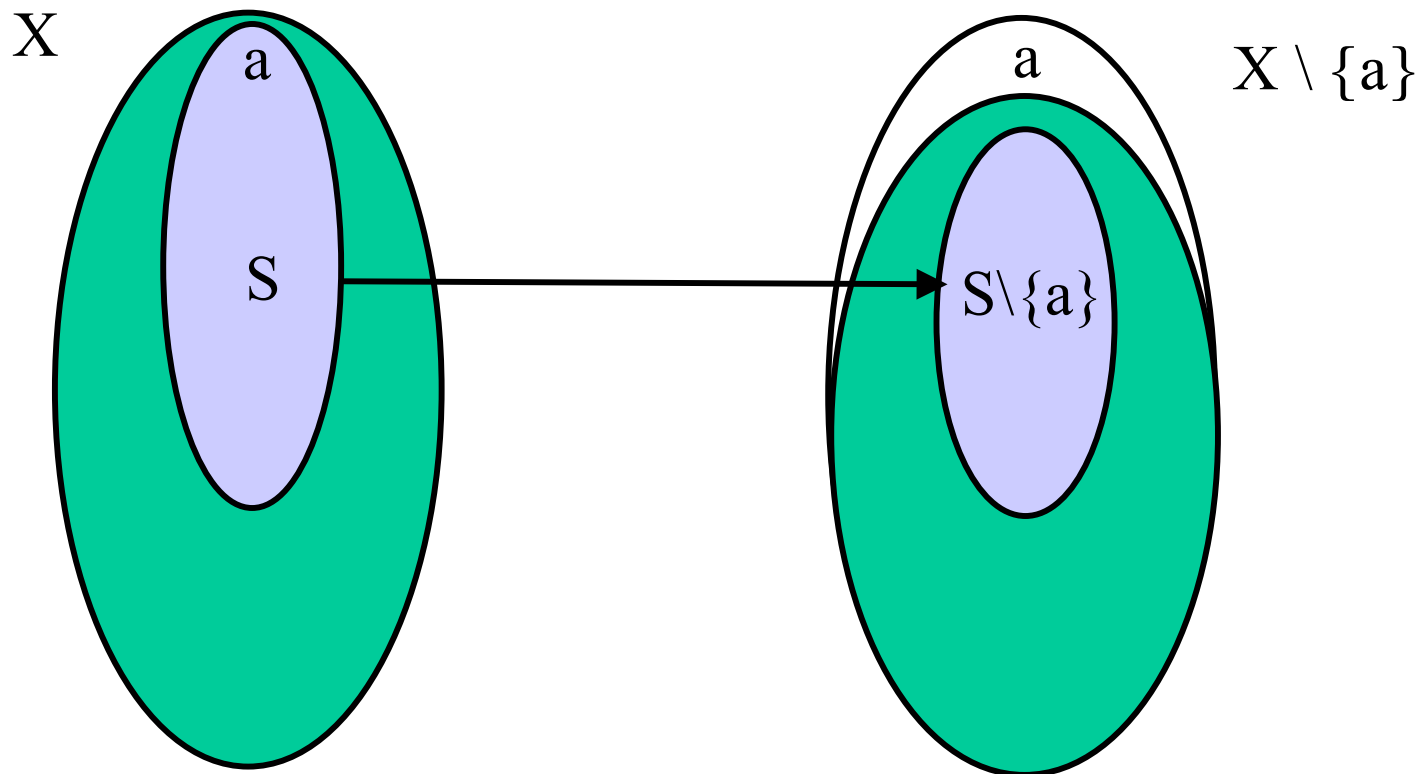
$$f(S) = b$$

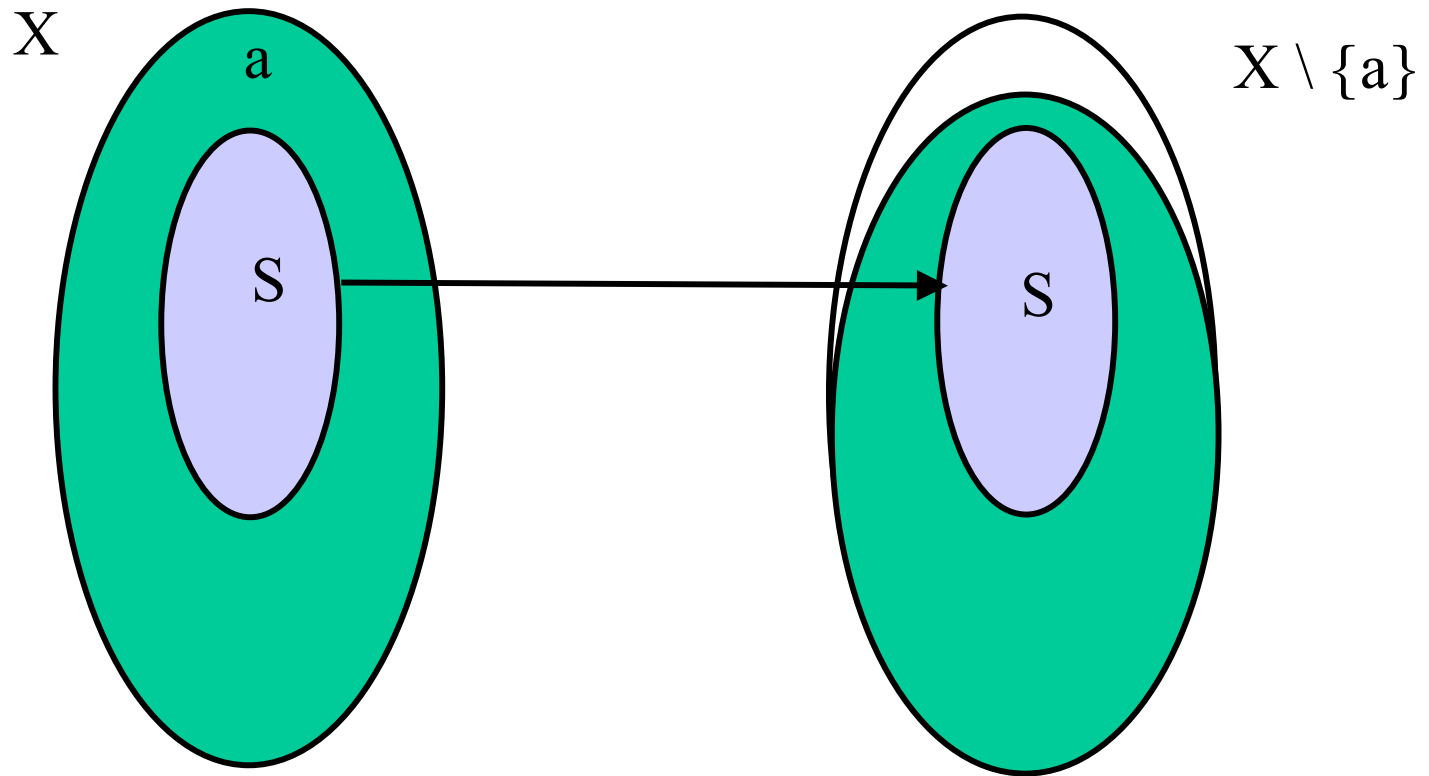
- **Bijection** between subsets of an n element set and bit sequences of length n
- f is 1-1
- f is onto

- Number of bit sequences of length n is 2^n
- Number of subsets of an n element set is 2^n

Odd and even

- Consider an n element set X
- How many subsets of odd size does X have ?
- Fix an element $a \in X$
- Bijection between **odd subsets** of X and **subsets** of $X \setminus \{a\}$

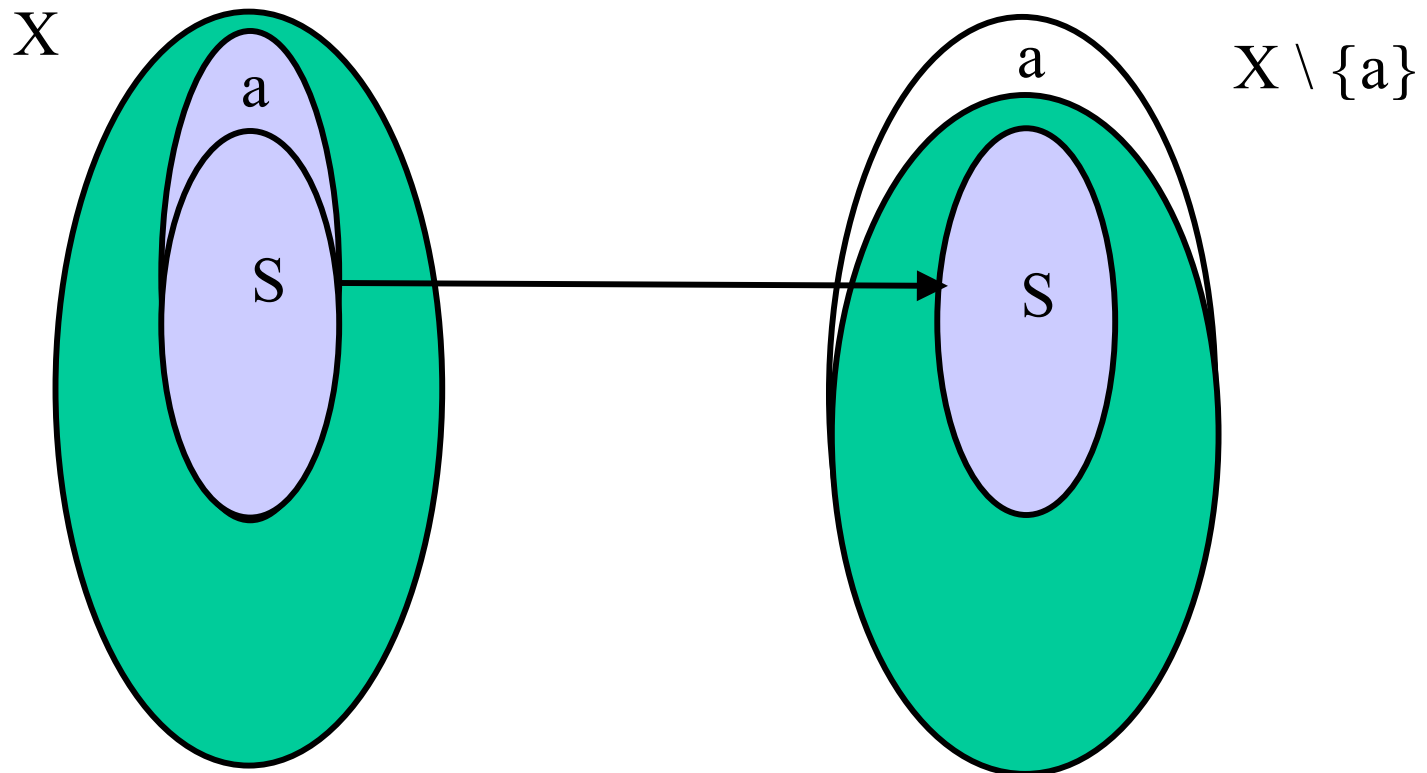




Hence number of odd subsets is exactly 2^{n-1}

Why is this correct ?

Can two different odd subsets map to the same subset of $X \setminus \{a\}$?



Counting with choices

- We want to count the number of objects in a set
- We give a process to select an object in the set by making a sequence of choices
 - Each distinct sequence of choices leads to an object in the set
 - Each object in the set can be produced by exactly one sequence of choices
- Suppose there are P_1 possibilities for 1st choice,
 P_2 possibilities for 2nd choice,
.... P_n possibilities for nth choice
- Then there are $P_1 \times P_2 \times \dots \times P_n$ objects in the set

Ordering a deck of cards

- How many different orderings of a deck of cards ?
- 52 choices for the 1st card
- 51 choices for the 2nd card
- 50 choices for the 3rd card
- ...
- 1 choice for the 52nd card
- Number of orderings = $52 \times 51 \times 50 \times \dots \times 1$
= $52 !$
(52 factorial)

Permutations

- A **permutation** or an **arrangement** of **n** objects is an ordering of the objects.
- The number of permutations of **n** distinct objects is **$n!$**

Permutations of r out of n

- The number of ways of ordering, permuting, or arranging r out of n objects.
- n choices for first place, $n-1$ choices for second place, . . .
- $n \times (n-1) \times (n-2) \times \dots \times (n-(r-1))$

$$= \frac{n!}{(n-r)!}$$

- How many sequences of 5 letters contain at least two of the same letter ?

Counting the opposite

- How many sequences of 5 letters contain at least two of the same letter ?

= no. 5-letter sequences

– no. 5-letter sequences with all letters distinct

$$= 26^5 - 26 \times 25 \times 24 \times 23 \times 22$$

Ordered Versus Unordered

From a deck of 52 cards how many **ordered** pairs can be formed?

$$- 52 \times 51$$

How many **unordered** pairs?

- $52 \times 51 / 2$ ← divide by overcount

Each unordered pair is listed twice on a list of the ordered pairs, but we consider the ordered pairs to be the same.

Ordered Versus Unordered

- From a deck of 52 cards how many **ordered** 5 card sequences can be formed?
 - $52 \times 51 \times 50 \times 49 \times 48$
- How many orderings of 5 cards?
 - $5!$
- How many **unordered** 5 card hands?

$$52 \times 51 \times 50 \times 49 \times 48 / 5! = 2,598,960$$

A combination or choice of r out of n objects is an (unordered) set of r of the n objects.

- The number of r combinations of n objects:

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

n choose r

