Introduction to Probability

COS 341 Fall 2002, lectures 20-22

Basic Laws of Probability

Definition 1 *A* sample space S *is a nonempty set whose elements are called* outcomes*. The* events *are subsets of* S*.*

Since events are subsets, we can apply the usual set operations to events to obtain new events. For events A and B, the event $A \cap B$ represents the set of outcomes that are in both event A and event B, i.e. $A \cap B$ represents the event A *and* B. Similarly, $A \cup B$ represents the event A *or* B.

Definition 2 *A* probability space *consists of a sample space* S *and a probability function* $\mathbf{Pr}()$*, mapping the events of* S *to real numbers in* $[0, 1]$*, such that:*

1. $\mathbf{Pr}(S) = 1$*, and*
2. *If* $A_0, A_1, \ldots$ *is a sequence of disjoint events, then*

$$\mathbf{Pr}\left(\bigcup_{i \in \mathbb{N}}\right) = \sum_{i \in \mathbb{N}} \mathbf{Pr}(A_i). \qquad \textit{(Sum Rule)}$$

One consequence of this definition is the following:

$$\mathbf{Pr}(\overline{A}) = 1 - \mathbf{Pr}(\overline{A}). \qquad \text{(Complement Rule)}$$

Several basic rules of probability parallel facts about cardinalities of finite sets:

$$\begin{aligned} \mathbf{Pr}(B - A) &= \mathbf{Pr}(B) - \mathbf{Pr}(A \cap B) && \text{(Difference Rule)} \\ \mathbf{Pr}(A \cup B) &= \mathbf{Pr}(A) + \mathbf{Pr}(B) - \mathbf{Pr}(A \cap B) && \text{(Inclusion-Exclusion)} \end{aligned}$$

An immediate consequence of (Inclusion-Exclusion) is

$$\mathbf{Pr}(A \cup B) \le \mathbf{Pr}(A) + \mathbf{Pr}(B) \qquad \text{(Boole's Inequality)}$$

Similarly (Difference Rule) impies that

$$\text{If } A \subseteq B, \text{ then } \mathbf{Pr}(A) \le \mathbf{Pr}(B). \qquad \text{(Monotonicity)}$$

Example 1 *Suppose we wire up a circuit containing a total of* n *connections. The probability of getting any one connection wrong is* p*. What can we say about the probability of wiring the circuit correctly ? (The circuit is wired correctly iff all the* n *connections are made correctly.)*

solution: Let A_i denote the event that connection i is made *correctly.* So $\mathbf{Pr}(\overline{A_i}) = p$.

$$\mathbf{Pr}(\text{all connections correct}) = \mathbf{Pr}\left(\cap_{i=1}^{n} A_i\right).$$

Without any additional assumptions (on the dependence of the events A_i), we cannot get an exact answer. However, we can give reasonable upper and lower bounds.

$$\mathbf{Pr}\left(\cap_{i=1}^{n} A_i\right) \leq \mathbf{Pr}(A_i) = 1 - p$$

$$\mathbf{Pr}\left(\cap_{i=1}^{n} A_i\right) = 1 - \mathbf{Pr}\left(\overline{\cap_{i=1}^{n} A_i}\right) = 1 - \mathbf{Pr}\left(\cup_{i=1}^{n} \overline{A_i}\right) \geq 1 - \sum_{i=1}^{n} \mathbf{Pr}(\overline{A_i}) = 1 - np$$

Both these bounds are tight, i.e. we can construct situations where the correct answer is equal to the upper bound and those where the correct answer is equal to the lower bound. ■

Conditional Probability

Definition 3 $\mathbf{Pr}(A|B)$ *denotes the probability of event A given that event B has occured.*

$$\mathbf{Pr}(A|B) ::= \frac{\mathbf{Pr}(A \cap B)}{\mathbf{Pr}(B)}$$

provided $\mathbf{Pr}(B) \neq 0$.

Rearranging terms gives the following:

Rule 1 (Product rule, base case) *Let A and B be events, with* $\mathbf{Pr}(B) \neq 0$. *Then*

$$\mathbf{Pr}(A \cap B) = \mathbf{Pr}(B) \cdot \mathbf{Pr}(A|B).$$

Rule 2 (Product rule, general case) *Let $A_1, A_2, \ldots, A_n$ be events.*

$$\mathbf{Pr}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbf{Pr}(A_1) \cdot \mathbf{Pr}(A_2|A_1) \cdot \mathbf{Pr}(A_3|A_1 \cap A_2) \cdot \ldots \cdot \mathbf{Pr}(A_n|A_1 \cap \cdots \cap A_{n-1}).$$

Case Analysis

Theorem 1 (Total Probability) *If a sample space is the disjoint union of events $B_1, B_2, \ldots$, then for all events A,*

$$\mathbf{Pr}(A) = \sum_{i \in \mathbb{N}} \mathbf{Pr}(A \cap B_i).$$

Corollary 1 (Total Probability) *If a sample space is the disjoint union of events $B_1, B_2, \ldots$, then for all events A,*

$$\mathbf{Pr}(A) = \sum_{i \in \mathbb{N}} \mathbf{Pr}(A|B_i)\, \mathbf{Pr}(B_i).$$

Independence

Definition 4 *Suppose A and B are events, and B has positive probability. Then A is* independent *of B iff*

$$\mathbf{Pr}(A|B) = \mathbf{Pr}(A)$$

The above definition does not apply when $\mathbf{Pr}(B) = 0$. We will extend the definition to the zero probability case as follows:

Definition 5 *If A and B are events and $\mathbf{Pr}(B) = 0$, then A is defined to be independent of B.*

Now we can define independence in an alternate way:

Theorem 2 *Events A and B are independent iff*

$$\mathbf{Pr}(A \cap B) = \mathbf{Pr}(A) \cdot \mathbf{Pr}(B). \qquad \textit{(Independence Product Rule)}$$

Note that *disjoint* events are not the same as *independent* events. In general disjoint events are not independent.