Graph Theory: Matchings and Hall’s Theorem

COS 341 Fall 2002, lecture 19

Definition 1 A matching $M$ in a graph $G(V, E)$ is a subset of the edge set $E$ such that no two edges in $M$ are incident on the same vertex.

The size of a matching $M$ is the number of edges in $M$. For a graph $G(V, E)$, a matching of maximum size is called a maximum matching.

Definition 2 If $M$ is a matching in a graph $G$, a vertex $v$ is said to be $M$-saturated if there is an edge in $M$ incident on $v$. Vertex $v$ is said to be $M$-unsaturated if there is no edge in $M$ incident on $v$.

If $G(V_1, V_2, E)$ is a bipartite graph than a matching $M$ of $G$ that saturates all the vertices in $V_1$ is called a complete matching.

Definition 3 Given a matching $M$ in graph $G$, an $M$-alternating path (cycle) is a path (cycle) in $G$ whose edges are alternately in $M$ and outside of $M$ (i.e. if an edge of the path is in $M$, the next edge is outside $M$ and vice versa). An $M$-alternating path whose end vertices are $M$-unsaturated is called an $M$-augmenting path.

Lemma 1 If $M$ is a maximum matching in a graph $G(V, E)$, there can be no $M$-augmenting paths in $G$.

Proof: Assume, for contradiction, that there exists an $M$-augmenting path $P$. Then we can flip the edges of $P$ to obtain a new matching by removing the edges of $P \cap M$ and adding the edges of $P \cap \overline{M}$. More formally, we set $M' = M \cup (P \cap \overline{M}) \setminus (P \cap M)$. It is easy to verify that $M'$ is indeed a valid matching in $G$. Further, $|M'| = |M| + 1$. This contradicts the fact that $M$ is a maximum matching.

Given a bipartite graph $G(V_1, V_2, E)$, and a subset of vertices $S \subseteq V_1$, the neighborhood $N(S)$ is the subset of vertices of $V_2$ that are adjacent to some vertex in $S$, i.e.

$$N(S) = \{ v \in V_2 : \exists u \in S, (u, v) \in E \}$$

Theorem 1 (Hall’s Theorem) Let $G(V_1, V_2, E)$ be a bipartite graph with $|V_1| \leq |V_2|$. Then $G$ has a complete matching saturating every vertex of $V_1$ iff $|S| \leq |N(S)|$ for every subset $S \subseteq V_1$. 

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Proof: First we prove that the condition of the theorem is necessary. If $G$ has a complete matching $M$ and $S$ is any subset of $V_1$, every vertex in $S$ is matched by $M$ into a different vertex in $N(S)$, so that $|S| \leq |N(S)|$.

Now we prove that the condition is sufficient. Suppose that $|S| \leq |N(S)|$ for every subset $S \subseteq V_1$. Assume for contradiction that $G$ has no complete matching. Let $M$ be a maximum matching, i.e. a matching that saturates the maximum number of vertices in $V_1$. Since $M$ is not complete, there exists an $M$-unsaturated vertex $s$ in $V_1$. Let $Z$ be the set of vertices of $G$ reachable from $s$ by $M$-alternating paths. Since $M$ is a maximum matching, there are no $M$-augmenting paths among these (by Lemma 1). Let $S = Z \cap V_1$ and $T = Z \cap V_2$. Then, every vertex of $T$ is matched under $M$ to some vertex of $S - \{s\}$ and every vertex of $S - \{s\}$ is matched under $M$ to some vertex of $T$. Thus $|T| = |S| - 1$. Also, $T = N(S)$. Thus $S$ is a subset of $V_1$ such that $|N(S)| = |S| - 1$, giving a contradiction. This proves the reverse direction. 

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