COS 341 – Discrete Math

Office Hours

- Currently, my office hours are on Friday, from 2:30 to 3:30.

Office Hours

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- Nobody seems to care.
- Change office hours? Tuesday, 8 PM to 9 PM.

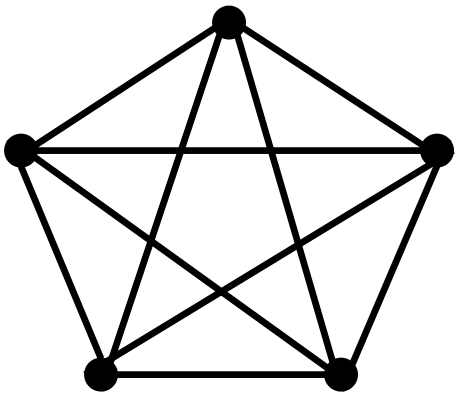
Homework 8

- Due on Wednesday **at the beginning of class**.
- No collaboration!

- Question 3:
  - "Never crosses itself" is the key.
- Question 4:
  - Assume $n > 4$ (the theorem is not true for n=4).
  - For some values of $n > 4$, the bound may not be an integer. It doesn't matter (the number of crossings will be strictly greater than that).

From last class

- Jordan curve theorem:
  - Any Jordan curve divides the plane into two parts, the *interior* and the *exterior*.
- K_5 is not planar.
- $K_{3,3}$ is not planar.



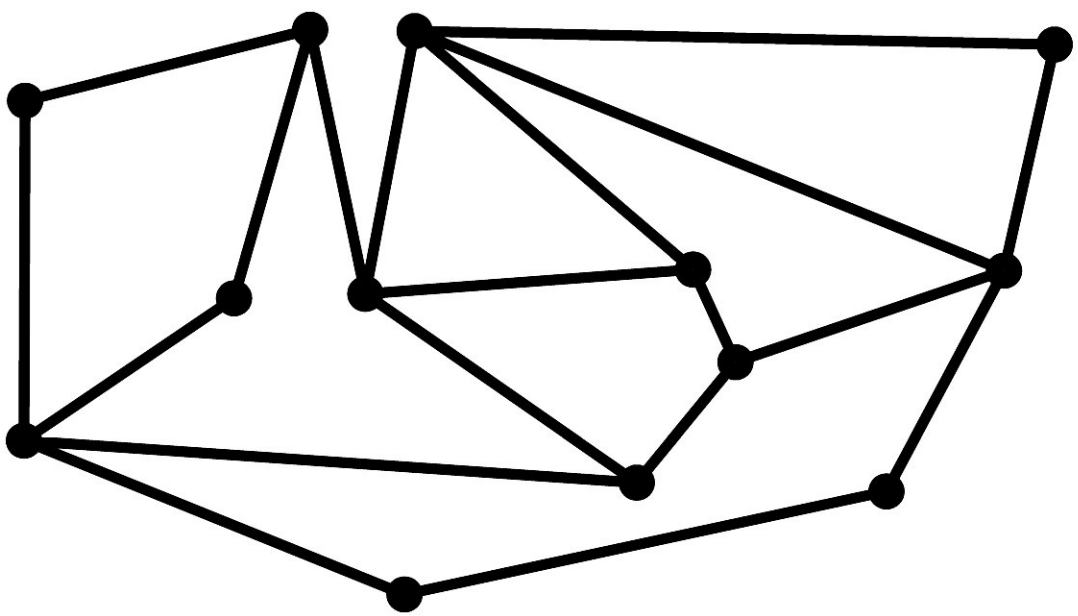
2-Connected Graphs

- Recall that a graph is 2-connected if it has at least 3 vertices, and by deleting any single vertex we obtain a connected graph.

- We also know the following:
  – *A graph G is 2-connected if and only if it can be created from a triangle (K_3) by a sequence of edge subdivisions and edge insertions.*

Faces and Cycles

- Theorem:
  - *Let G be a 2-vertex-connected planar graph. Then every face in any planar drawing of G is a region of some cycle of G.*



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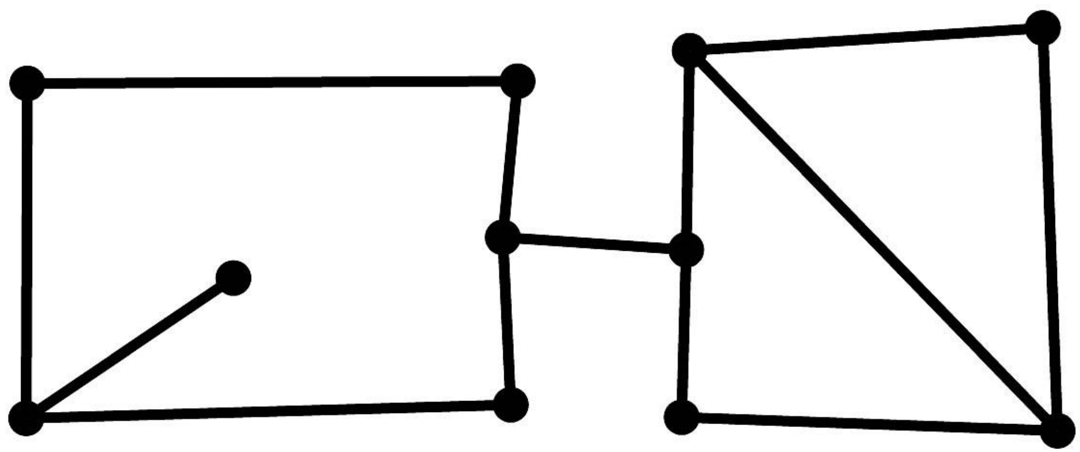


(We do need it to be 2-vertex-connected.)

Faces and Cycles

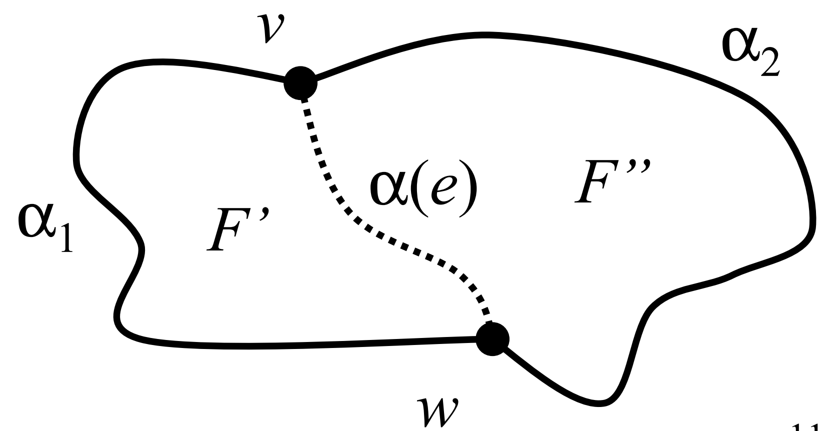
- Proof: by induction on *n* (number of vertices)
  - Base case: $n = 3$
    - only 2-connected graph is the triangle
    - one cycle, two regions: OK.
  - Hypothesis: assume true for $n = n_o - 1$, with $n_0 > 3$.
  - Let's prove it is true for $n = n_o$.
    - 2-connected graph *G* with at least 4 vertices.

Faces and Cycles

- Take a planar 2-connected graph G with $n > 3$ vertices.
- Can be built from a triangle by a sequence of edge insertions and subdivisions.
- One of these must be true:
  - (a) There is an edge e such that $G' = G - e$ is 2-connected.
  - (b) There is a graph $G' = (V', E')$ and there is an edge e' in E' such that the subdivision of e' creates G.
- In either case, G' is a smaller 2-connected graph.
  - By the inductive hypothesis, every face in any planar drawing of G' is a region of some cycle of G'.

Faces and Cycles

– Case (a): there is an edge e such that $G' = G - e$ is 2-connected.
  - Let $e = \{v,w\}$.
  - There is a face F in G' corresponding to a cycle that contains both v and w.
    - $v - \alpha_1 - w - \alpha_2 - v$ (α_1 and α_2 are arcs in the cycle)
  - The arc corresponding to e divides F into two faces, each corresponding to a different cycle.
    - $v - \alpha_1 - w - \alpha(e) - v$
    - $v - e - w - \alpha_2 - v$



Faces and Cycles

– Case (b): there is a graph *G'* = (*V*, *E'*) with an edge *e'* in *E'* such that the subdivision of *e'* creates *G*.
  - Each face of *G'* is a region of some cycle *G'*.
  - Subdividing *e'* amounts to drawing a vertex inside the edge.
  - This extends the length of the cycles *e'* participates in, but doesn't change the property.

Combinatorial Characterization

Combinatorial Characterization

- Every subgraph of a planar graph must be planar:
  - cannot contain K_5
  - cannot contain $K_{3,3}$

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Combinatorial Characterization

- Every subgraph of a planar graph must be planar:
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- More generally: no subgraph of a planar graph can be a subdivision of a non-planar graph.
  - cannot contain a subdivision of K_5
  - cannot contain a subdivision of $K_{3,3}$
- Is that enough?

Combinatorial Characterization

- Kuratowski's theorem:
  - *A graph G is planar if and only if it has no subgraph isomorphic to a subdivision of* $K_{3,3}$ *or to a subdivision of* K_5.
- We can test if a graph is planar without actually drawing it:
  - we just have to verify if there are violating subgraphs.
  - (There are faster ways of testing planarity, though.)

Euler's Formula

- Theorem:
  - *Let G = (V, E) be a connected planar graph, and let f be the number of faces of any planar drawing of G. Then*

$$|V| - |E| + f = 2.$$

- The number of faces does not depend on the (planar) drawing, just on the graph itself.

Euler's Formula

- Proof by induction on $|E|$.
  - Base case: $|E| = 0$ (single vertex, single face):

$$|V| - |E| + f = 1 - 0 + 1 = 2.$$

  - $|E| > 0$ and G does not contain a cycle (it's a tree):

$$|V| - |E| + f = |V| - (|V| - 1) + 1 = 2.$$

  - $|E| > 0$ and $G = (V, E)$ contains a cycle:
    - Some edge e belongs to a cycle; remove it.
    - The resulting graph G' obeys the formula: $|V'| - |E'| + f' = 2$
      - Clearly, $|V'| = |V|$ and $|E'| = |E| - 1$.
      - e was adjacent to two faces (by Jordan) that become one: $f' = f - 1$

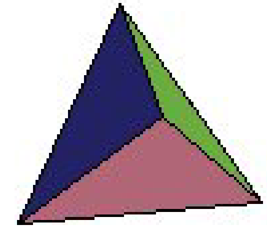
$$|V'| - |E'| + f' = 2$$

$$|V| - (|E| - 1) + (f - 1) = |V| - |E| + f = 2.$$

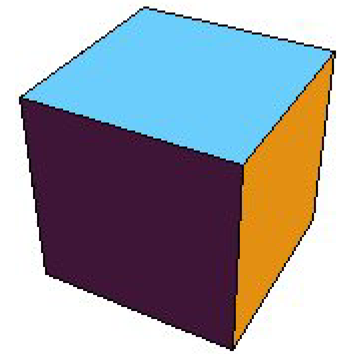
Regular Polytopes

- 3-dimensional convex bodies;
- finite number of faces;
- faces are congruent copies of the same regular polygon;
- same number of faces meet at each vertex;
- also known as *Platonic Solids*.

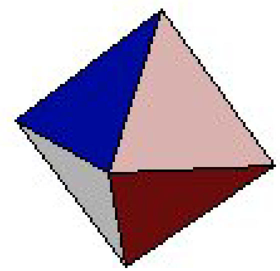
Regular Polytopes



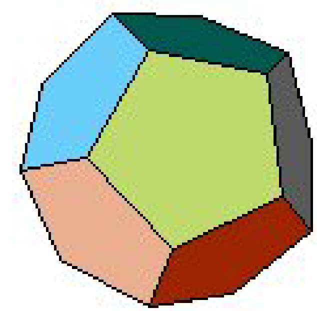
- Tetrahedron: 4 faces
- Hexahedron (a.k.a. cube): 6 faces



- Octahedron: 8 faces
- Dodecahedron: 12 faces

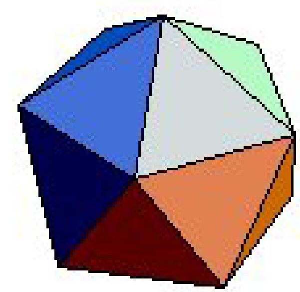


- Icosahedron: 20 faces



- Are there more?

[images from mathworld.wolfram.com]

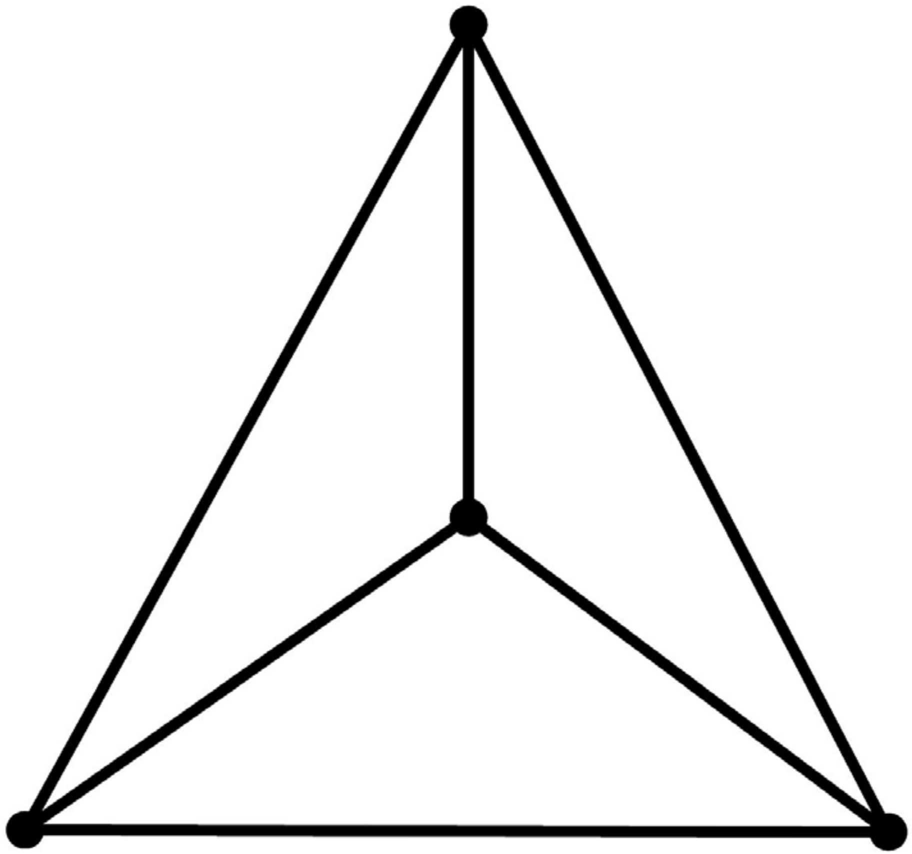


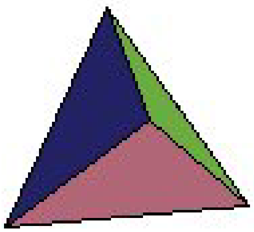
Regular Polytopes

- Every convex polytope can be converted to a planar graph:
  - Find a sphere such that:
    - center of sphere inside polytope;
    - sphere contains the whole polytope.
  - Project the polytope onto the sphere:
    - we get a graph of the surface of a sphere;
    - that graph can be converted to a planar graph with a stereographic projection.
  - Vertices, faces, and edges of the polytope become vertices, faces, and edges of a planar graph.

Regular Polytopes

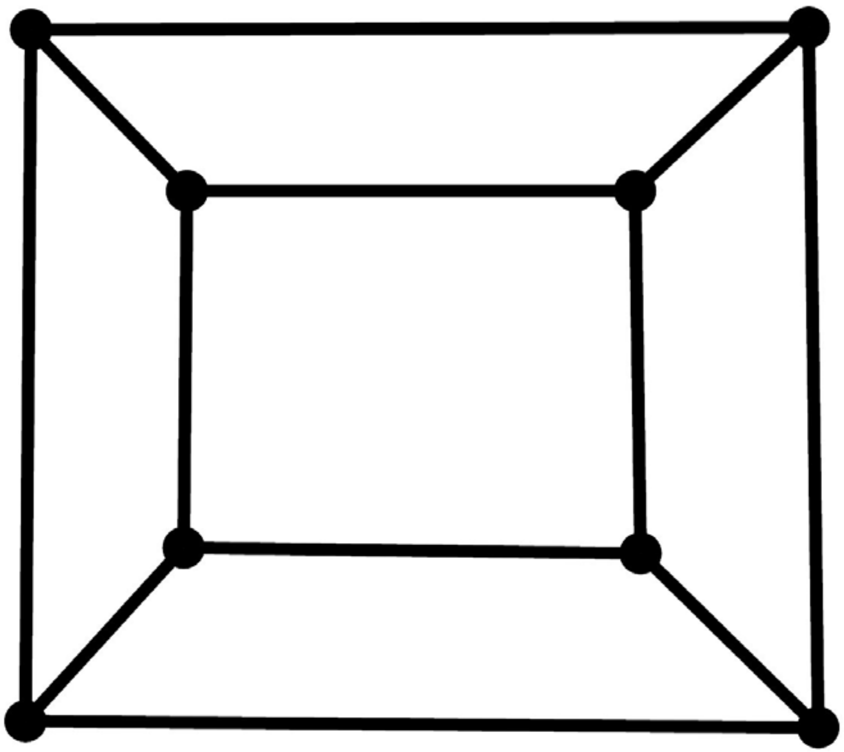
- Tetrahedron

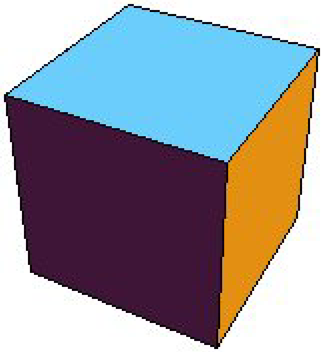




Regular Polytopes

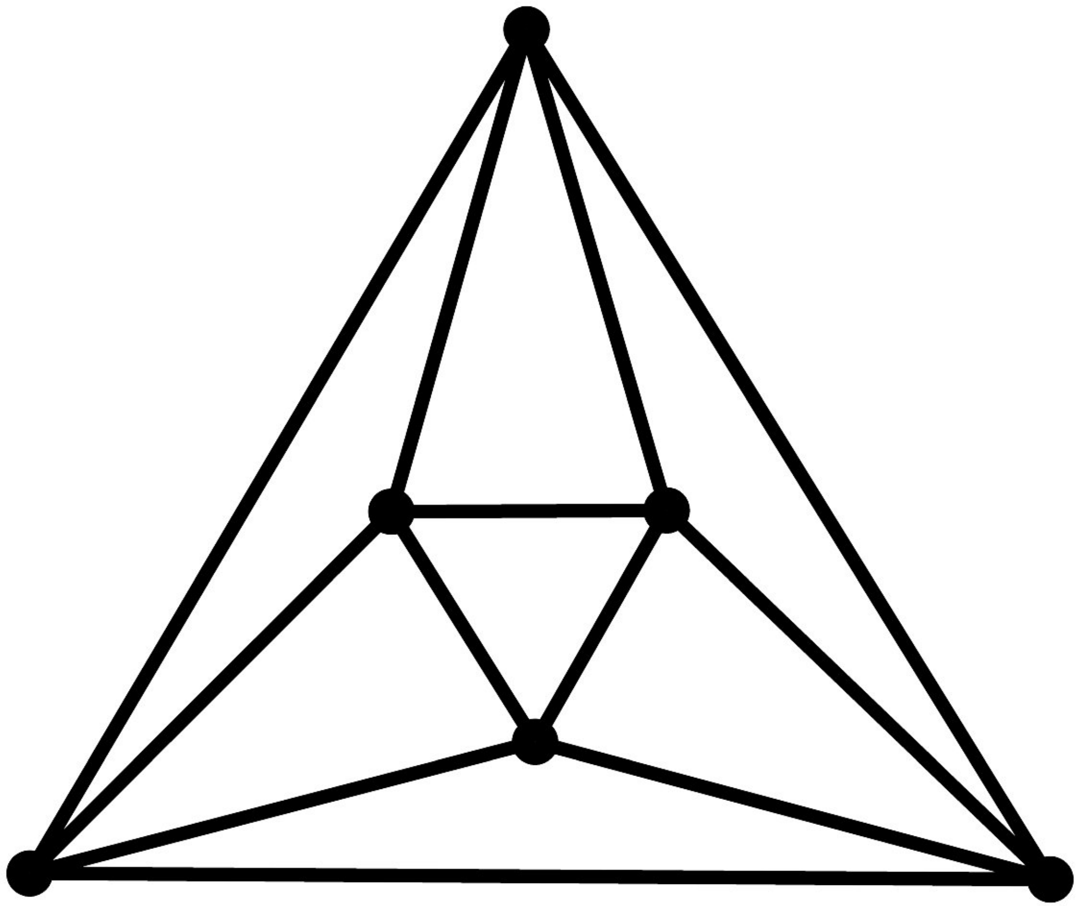
- Cube

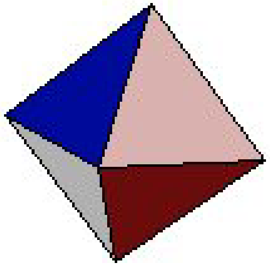




Regular Polytopes

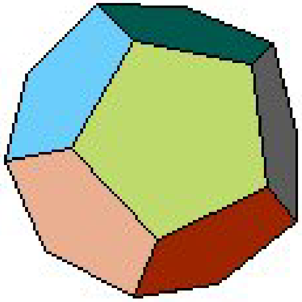
- Octahedron

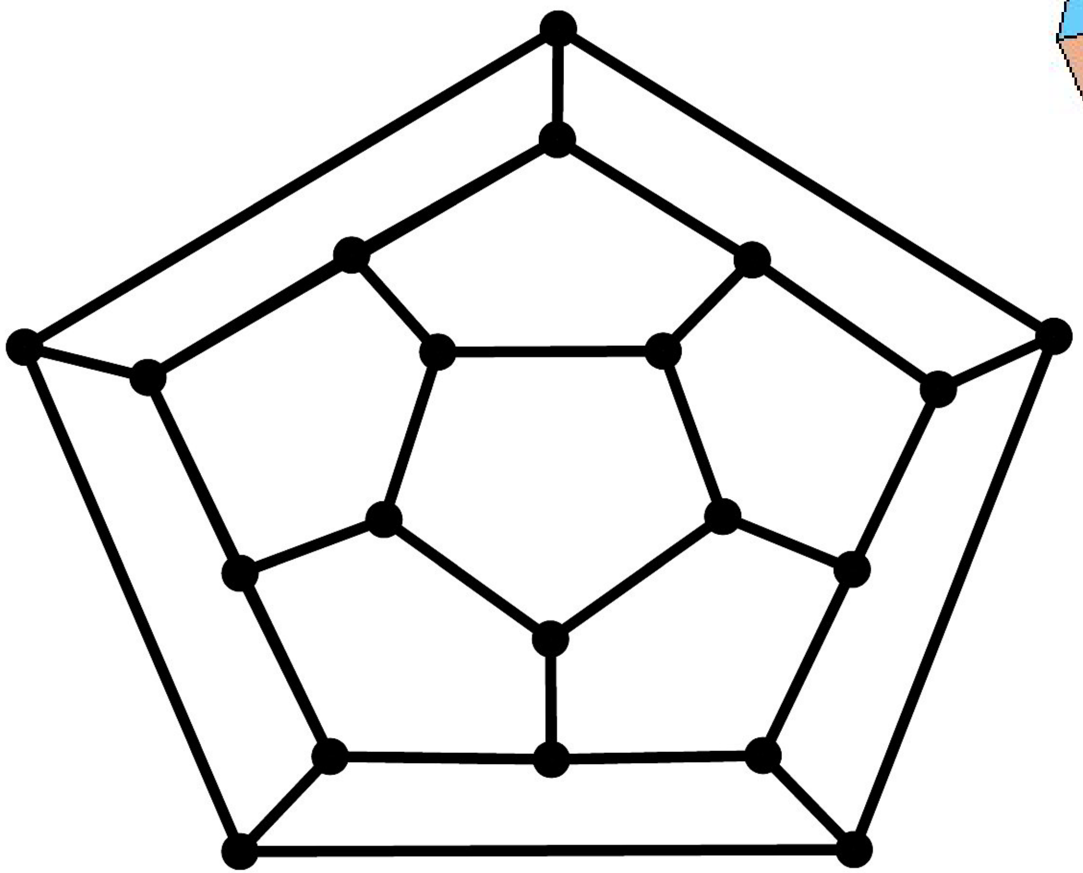




Regular Polytopes

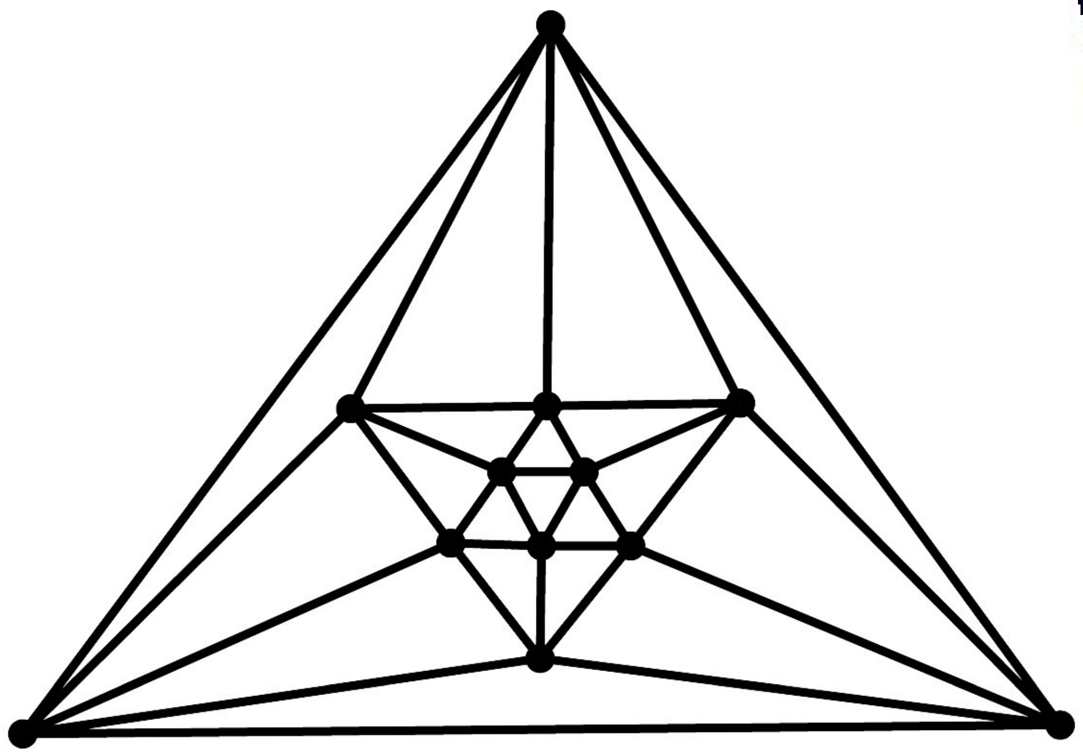
- Dodecahedron

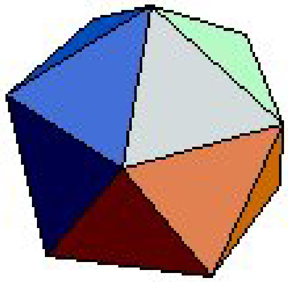




Regular Polytopes

- Icosahedron





Regular Polytopes

- Parameters of a regular convex polytope:
  - k: number of sides in each polygon (face)
  - d: number of faces that meet at each vertex
  - n: vertices
  - m: edges
  - f: faces
- Looking at the vertices:
  - Every edge appears in exactly two vertices:

$$dn = 2m$$

$$n = 2m/d$$

Regular Polytopes

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Regular Polytopes

- Parameters of a regular convex polytopes:
  - k: number of sides in each face
  - d: number of faces that meet at each vertex
  - n: vertices
  - m: edges
  - f: faces
- Looking at the whole graph:
  - It is planar, so we can apply Euler's formula:

$$n - m + f = 2$$

Regular Polytopes

- So we have:

$$f = 2m/k$$

$$n = 2m/d$$

$$n - m + f = 2$$

- Substituting *n* and *f* in the third equation:

$$n - m + f = 2$$

$$2m/d - m + 2m/k = 2$$

(dividing by $2m$ and rearranging...)

$$\frac{1}{d} + \frac{1}{k} = \frac{1}{2} + \frac{1}{m}$$

Regular Polytopes

- So every regular polytope must obey $\frac{1}{d}+\frac{1}{k}=\frac{1}{2}+\frac{1}{m}$
- In particular,

$$\frac{1}{d}+\frac{1}{k}>\frac{1}{2}$$

- If both $d \geq 4$ and $k \geq 4$, we would have:

$$\frac{1}{d}+\frac{1}{k}\leq\frac{1}{2}$$

- Se either d=3 or k=*3* (or both).

Regular Polytopes

- Assume d=3:

$$\frac{1}{d}+\frac{1}{k}=\frac{1}{2}+\frac{1}{m}$$

$$\frac{1}{3}+\frac{1}{k}=\frac{1}{2}+\frac{1}{m}$$

$$\frac{1}{k}-\frac{1}{6}=\frac{1}{m}$$

- The right-hand side is positive, so $k < 6$.
- $k = \{3, 4, 5\}$

Regular Polytopes

- Assume k=3:

$$\frac{1}{d}+\frac{1}{k}=\frac{1}{2}+\frac{1}{m}$$

$$\frac{1}{d}+\frac{1}{3}=\frac{1}{2}+\frac{1}{m}$$

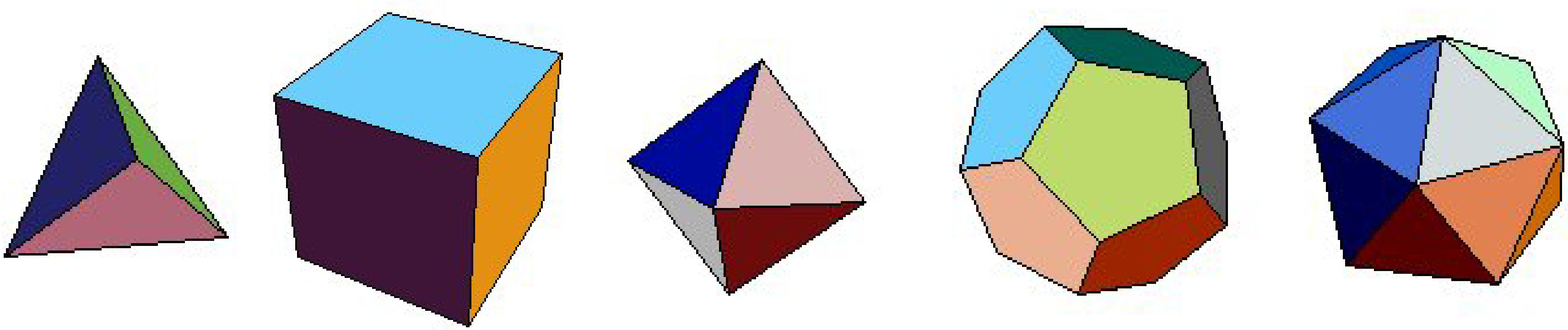
$$\frac{1}{d}-\frac{1}{6}=\frac{1}{m}$$

- The right-hand side is positive, so $d < 6$.
- $d = \{3, 4, 5\}$

Regular Polytopes

- So the only possibilities are:

d	*k*	*n*	*m*	*f*	Polytope
3	3	4	6	4	tetrahedron
3	4	8	12	6	cube
3	5	20	30	12	dodecahedron
4	3	6	12	8	octahedron
5	3	12	30	20	icosahedron



Number of Edges

- Theorem:
  - *Let G = (V,E) be a planar graph with at least 3 vertices. Then* $|E| \le 3|V| - 6$.
  - *If the graph is maximal (no edge can be added without violating planarity), the equality holds:* $|E| = 3|V| - 6$.

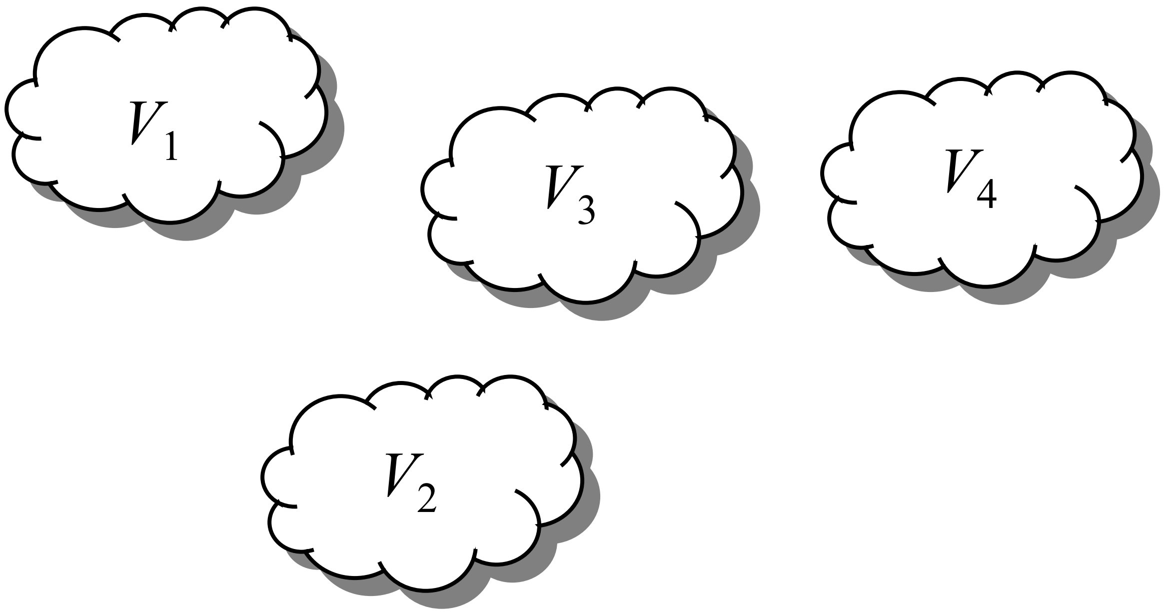
- It suffices to prove the second statement; if the graph is not maximal, we can always add edges until it becomes one.

Number of Edges

- Lemma:
  - *Every maximal planar graph G is a triangulation (every face is a triangle).*
- Proof: we show that if *G* is not a triangulation, it is always possible to add an edge without violating planarity.
  - Three cases to consider:
    - *G* is disconnected.
    - If *G* is connected but not 2-connected.
    - *G* is 2-connected.

Number of Edges

- Case 1: G is not connected:
  - An edge can be added between two components.



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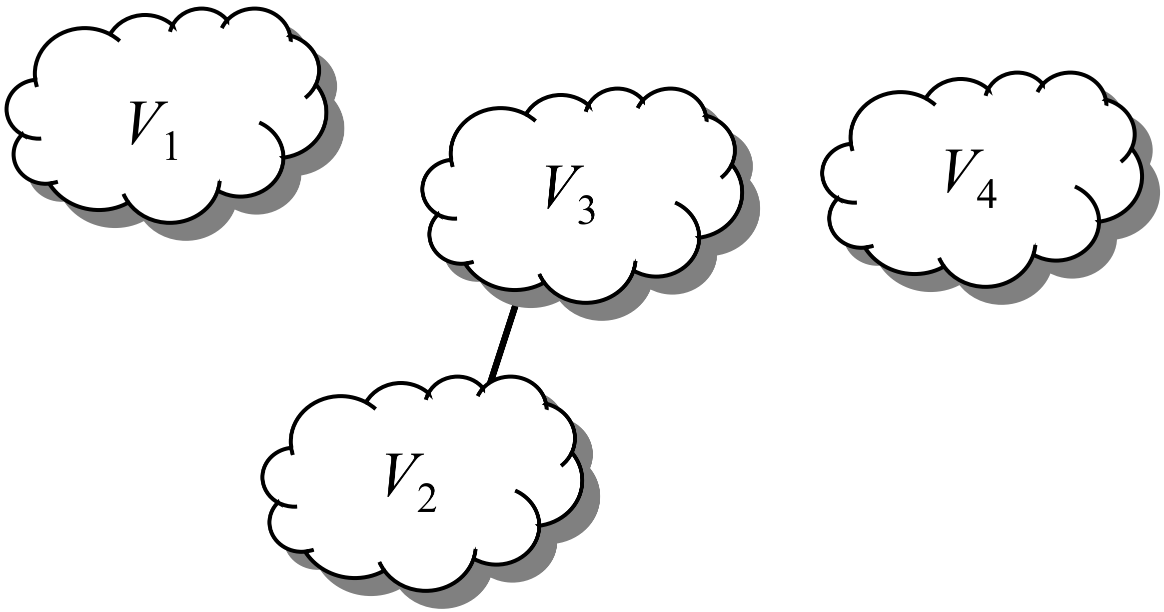
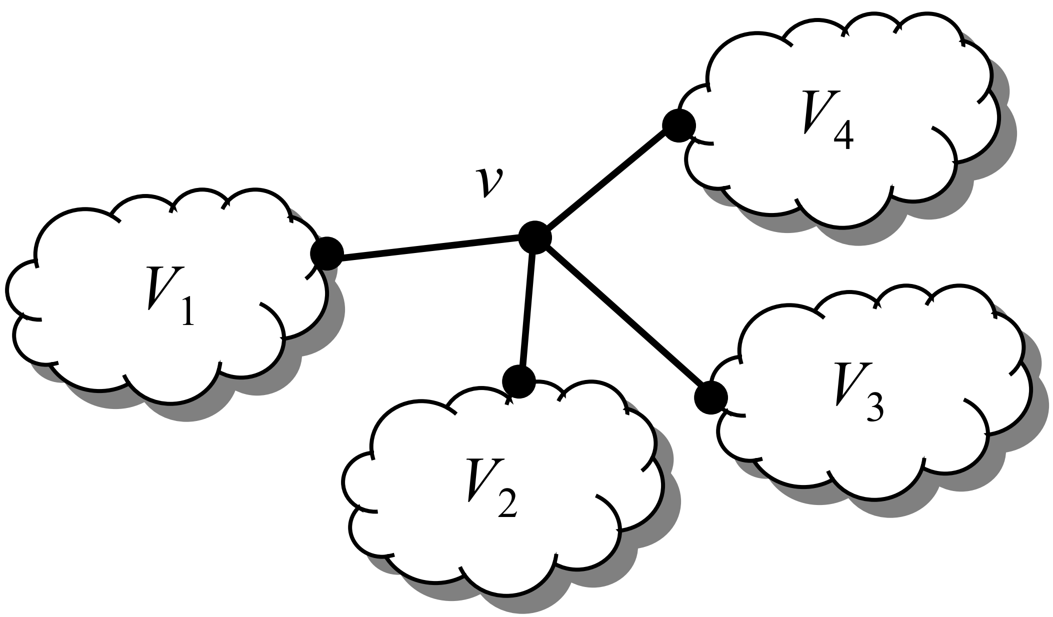


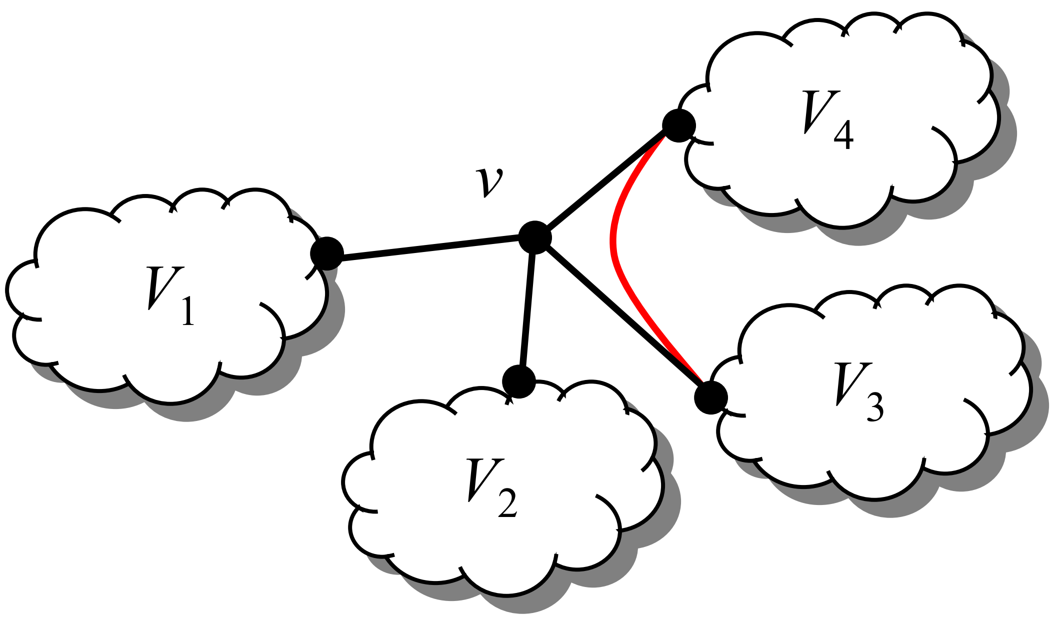
Number of Edges

- Case 2: *G* is connected, but not 2-connected:
  - There is a vertex *v* whose removal disconnects *G*.
  - Let V_1, V_2, ..., V_k be the resulting components ($k > 2$).
  - An edge can be added between components associated with edges drawn next to each other around *v*.



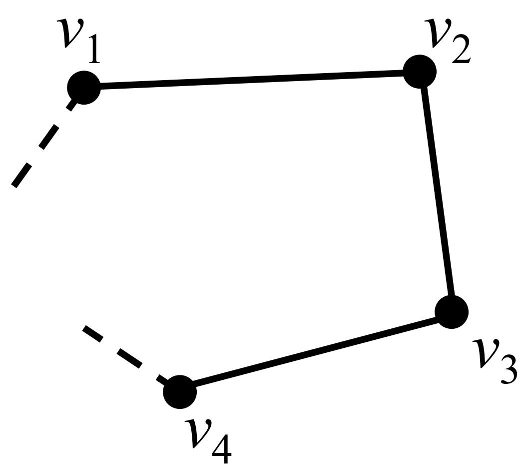
Number of Edges

- Case 2: G is connected, but not 2-connected:
  - There is a vertex v whose removal disconnects G.
  - Let V_1, V_2, ..., V_k be the resulting components ($k > 2$).
  - An edge can be added between components associated with edges drawn next to each other around v.



Number of Edges

- Case 3: G is 2-connected.
  - Every face is bounded by a cycle.
  - Take any face with 4 or more edges:



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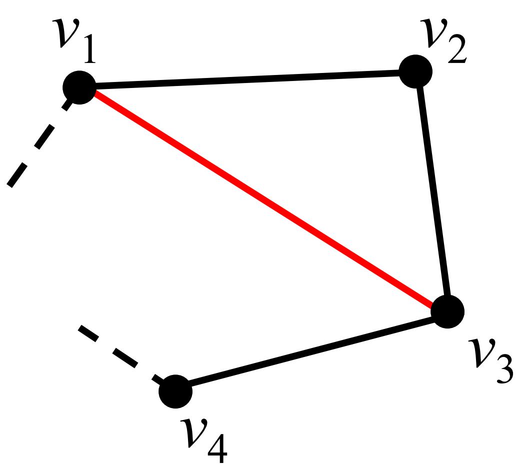
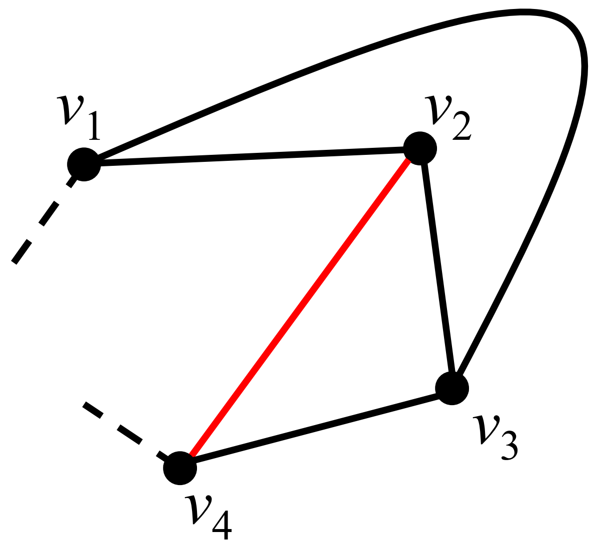


  - If v_1 and v_3 are not connected, you can add an edge between them.

Number of Edges

- Case 3: *G* is 2-connected.
  - Every face is bounded by a cycle.
  - Take any face with 4 or more edges:



  - If v_1 and v_3 are connected, v_2 and v_4 can't be.
  - So you can add an edge between v_2 and v_4.

Number of Edges

- So every maximal planar graph is a triangulation.
  - Because every face is a triangle and every edge is incident to exactly two faces, we have:

$$3f = 2|E|$$

$$f = 2|E|/3.$$

  - Using this value in Euler's formula:

$$|V| - |E| + f = 2$$

$$|V| - |E| + 2|E|/3 = 2$$

$$|V| - |E|/3 = 2$$

$$|E| = 3|V| - 6.$$

  - Corollary: there exists a vertex of degree at most 5.

Triangle-Free Planar Graphs

- Theorem:
  - *Let $G=(V,E)$ be a planar graph with no triangles (i.e., without K_3 as a subgraph) and at least 3 vertices. Then $|E| \leq 2|V| - 4$.*
- Proof (similar to the previous one)
  - Consider a maximal triangle-free planar graph G;
    - we can always add edges until it becomes one.
  - G is clearly connected.

Triangle-Free Planar Graphs

- Assume G is connected, but not 2-connected.
- There is a vertex v whose removal disconnects G.
- Let V_1, V_2, …, V_k be the resulting components ($k > 2$).
  - Edges can be added between these components without violating planarity.
  - But we could create a triangle if we joined vertices that are adjancent to v.
- If every V_i is a single vertex, then G is a tree:

$|E| = |V| - 1$

$|E| = |V| + 3 - 4$

$|E| \leq |V| + |V| - 4$ (because G has at least three vertices)

$|E| \leq 2|V| - 4$ (the inequality holds)

Triangle-Free Planar Graphs

- Now consider the case in which component V_1 has at least two vertices.
- Consider a face F having both a vertex of V_1 and a vertex of some other V_i on its boundary.
- V_1 must have at least one edge $\{v_1,v_2\}$ on the boundary of F.
- We can't have both v_1 and v_2 connected to v (or these vertices would constitute a triangle).
- So an edge can be added between one of these vertices and a vertex in V_i.
  - G is not maximal – a contradiction.
  - Maximal triangle-free planar graphs must be 2-connected.

Triangle-Free Planar Graphs

- G is a 2-connected, maximal triangle-free planar graph.
- 2-connected:
  - every face is a region of a cycle.
- Triangle-free:
  - every cycle has at least 4 edges.
- Counting edges from faces: $2\,|E| \geq 4f \Rightarrow f \leq |E|/2$
- From Euler's formula:

$$|V| - |E| + f = 2$$

$$2 - |V| + |E| = f \leq |E|/2$$

$$|E| \leq 2|V| - 4.$$

- Corollary: there exists a vertex of degree at most 3.

Scores of Planar Graphs

- Theorem:
  - *Let G=(V,E) be a 2-connected planar graph with at least 3 vertices. Define:*
    - n_i: *number of vertices of degree i;*
    - f_i: *number of faces (in some fixed drawing of G) bounded by cycles of length i.*

  Then we have

$$\sum_{i\geq 1}(6-i)n_i = 12 + 2\sum_{j\geq 3}(j-3)f_j .$$

Scores of Planar Graphs

- Why is this relevant?
- We can rewrite

$$\sum_{i\geq 1}(6-i)n_i = 12 + 2\sum_{j\geq 3}(j-3)f_j.$$

as

$$5n_1 + 4n_2 + 3n_3 + 2n_2 + n_1 + (\ldots) = 12 + (\ldots)$$

– The first "(...)" contains only negative terms.

– The second "(...)" contains only positive terms.

– So $5n_1 + 4n_2 + 3n_3 + 2n_2 + n_1 \geq 12$.

– Among other things, this means that there are at least 3 vertices of degree at most 5 *in every planar graph*.

Scores of Planar Graphs

- Proof of the theorem:
  - Obvious facts:

$$f = \sum_j f_j \quad \text{and} \quad |V| = \sum_i n_i$$

  - From Euler's formula:

$$|V| - |E| + f = 2$$

$$\sum_i n_i - |E| + \sum_j f_j = 2$$

$$2|E| = \sum_i 2n_i + \sum_j 2f_j - 4$$

Scores of Planar Graphs

- Proof of the theorem:
  - From previous slide: $2|E| = \sum_i 2n_i + \sum_j 2f_j - 4$
  - Counting edges from the faces:

$$\sum_j (j \cdot f_j) = 2|E| = \sum_i 2n_i + \sum_j 2f_j - 4$$

$$\sum_j (j \cdot f_j) - \sum_j 2f_j + 4 = \sum_i 2n_i$$

$$\sum_j (j-2)f_j + 4 = \sum_i 2n_i$$

  - Counting edges from the vertices:

$$\sum_i (i \cdot n_i) = 2|E| = \sum_i 2n_i + \sum_j 2f_j - 4$$

$$\sum_j 2f_j = \sum_i (i \cdot n_i) - \sum_i 2n_i + 4$$

$$\sum_j 2f_j = \sum_i n_i(i-2) + 4$$

Scores of Planar Graphs

- Proof of the theorem:
  - From the previous slide:

$$\sum_j (j-2) f_j + 4 = \sum_i 2n_i \qquad (\times 2)$$

$$\sum_j (2j \cdot f_j - 4 f_j) + 8 = \sum_i 4n_i \qquad \text{(i)}$$

$$\sum_j 2 f_j = \sum_i n_i (i-2) + 4 \qquad (\times (-1))$$

$$\sum_j (-2) f_j = \sum_i (2n_i - i \cdot n_i) - 4 \qquad \text{(ii)}$$

  - Adding (i) and (ii), we get the final expression:

$$\sum_j (2j \cdot f_j - 4 f_j - 2 f_j) + 8 = \sum_i (4n_i + 2n_i - i \cdot n_i) - 4$$

$$2\sum_j (j-3) f_j + 12 = \sum_i (6-i) n_i$$