COS 341   Discrete Mathematics

Administrative Information

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Administrative Information

- Required text: Invitation to Discrete Mathematics
  Jiri Matousek and Jaroslav Nesetril
- Reference: Discrete Mathematics and its Applications
  Kenneth Rosen
- All handouts posted on web page
- Class mailing list
- Homeworks assigned every Wed., due in class next Wed.
- Grading:
  9-10 homeworks (80%), take-home final (20%)
- Collaboration policy

Discussion Sessions/Office Hours

- Discussion Sessions in addition to office hours (perhaps)
- Times announced next week on basis of student choices
- My office hours for next week: Tue, 2:00-4:00 pm

What is Discrete Mathematics ?

- Mathematics dealing with finite sets
- Topics: counting, combinatorics, graph theory, probability
- Goals:
  Develop mathematical maturity
  Foundation for advanced courses in Computer Science

- Flavor of questions:
- How many valid passwords on a computer system ?
- What is the probability of winning a lottery ?
- What is the shortest path between two cities ?

Toy problems as illustrations

- Puzzle:
  Three houses, three wells:
  Can we connect each house to each well by pathways so that no two pathways cross ?

- Real world problem:
  VLSI: Given placement of components of circuit on a board, is it possible to connect them along a board so that no two wires cross ?

Proof techniques

Evidence vs. Proof

$$p(n) = n^2 + n + 41$$

Claim: $\forall n \in \mathrm{N},\ p(n)$ is prime

Evidence

$p(0) = 41$ prime

$p(1) = 43$ prime

$p(2) = 47$ prime

$p(3) = 53$ prime

$\vdots$

$p(20) = 461$ prime looks promising!

$\vdots$

$p(39) = 1601$ prime must be true!

Only prime numbers ?

$$\forall n \in \mathrm{N},\ p(n) = n^2 + n + 41 \text{ is prime}$$

- Cannot be a coincidence
- Hypothesis must be true !

But it is NOT !

$$p(40) = 1681 \quad \text{is NOT PRIME}$$

Evidence vs. Proof

- Euler's conjecture (1769):

$$a^4 + b^4 + c^4 = d^4$$

has no solution for a, b, c, d positive integers

Counterexample: 218 years later by Noam Elkies:

$$95800^4 + 217519^4 + 414560^4 = 422481^4$$

Example courtesy: Prof. Albert R. Meyer's lecture slides for MIT course 6.042, Fall 02

Evidence vs. Proof

- Hypothesis:

$$313 \cdot (x^3 + y^3) = z^3$$

has no positive integer solution

False !

But smallest counterexample has

more than 1000 digits !

Example courtesy: Prof. Albert R. Meyer's lecture slides for MIT course 6.042, Fall 02

Mathematical Induction

- Find a general formula for $\sum_{i=0}^{n} 2^i$
-

$$2^0 = 1$$

$$2^0 + 2^1 = 3$$

$$2^0 + 2^1 + 2^2 = 7$$

$$2^0 + 2^1 + 2^2 + 2^3 = 15$$

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \quad ?$$

Mathematical Induction

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

1. (Basis step)
   $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ holds for $n = 0$

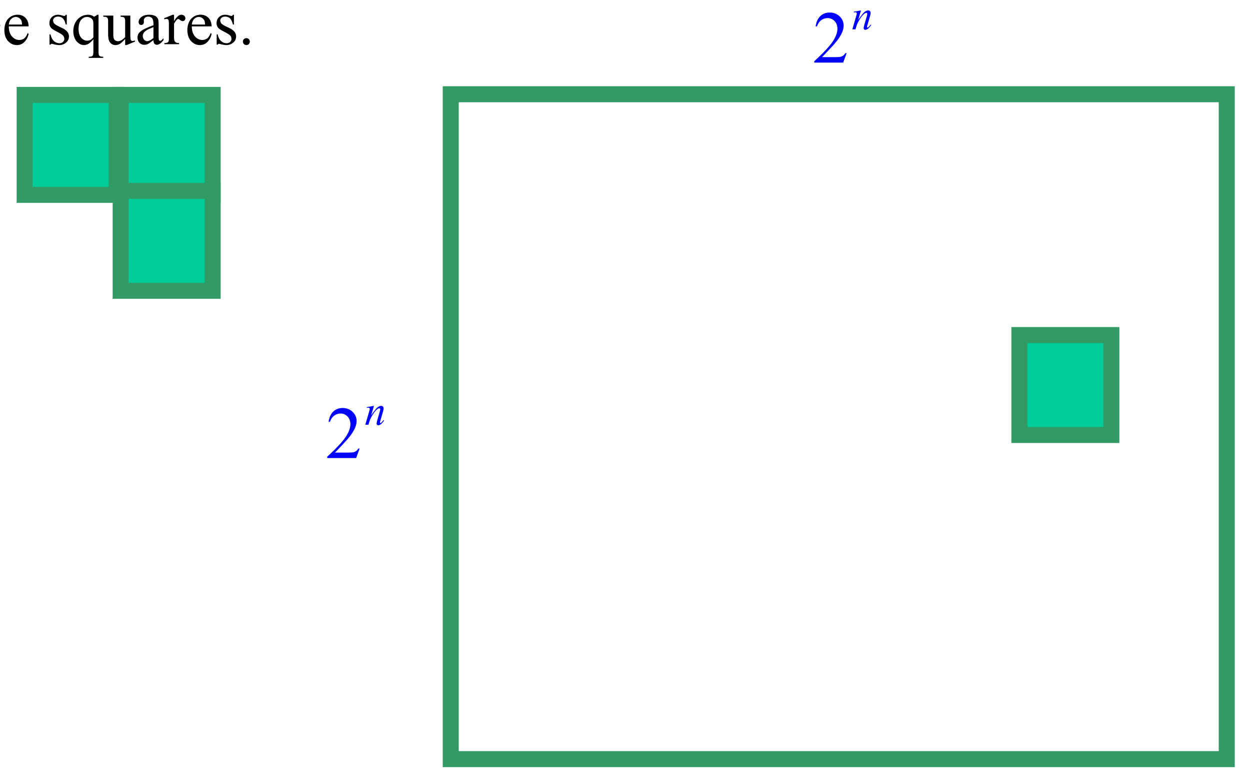
2. (Inductive step)
   Suppose formula holds for $n = n_0$ (inductive hypothesis)
   We prove that it also holds for $n = n_0 + 1$

$$\sum_{i=0}^{n_0+1} 2^i = \left(\sum_{i=0}^{n_0} 2^i \right) + 2^{n_0+1}$$

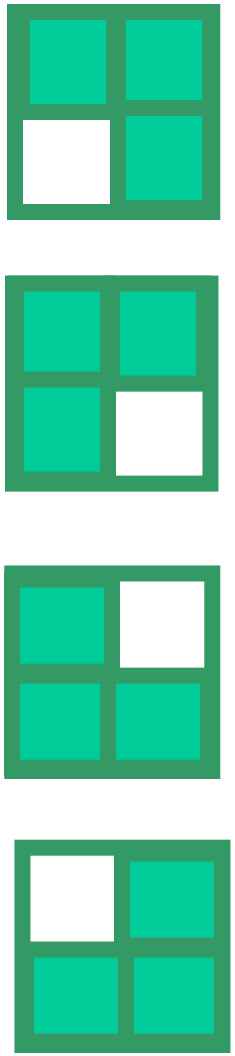
$$= \left(2^{n_0+1} - 1 \right) + 2^{n_0+1}$$

$$= 2^{n_0+2} - 1$$

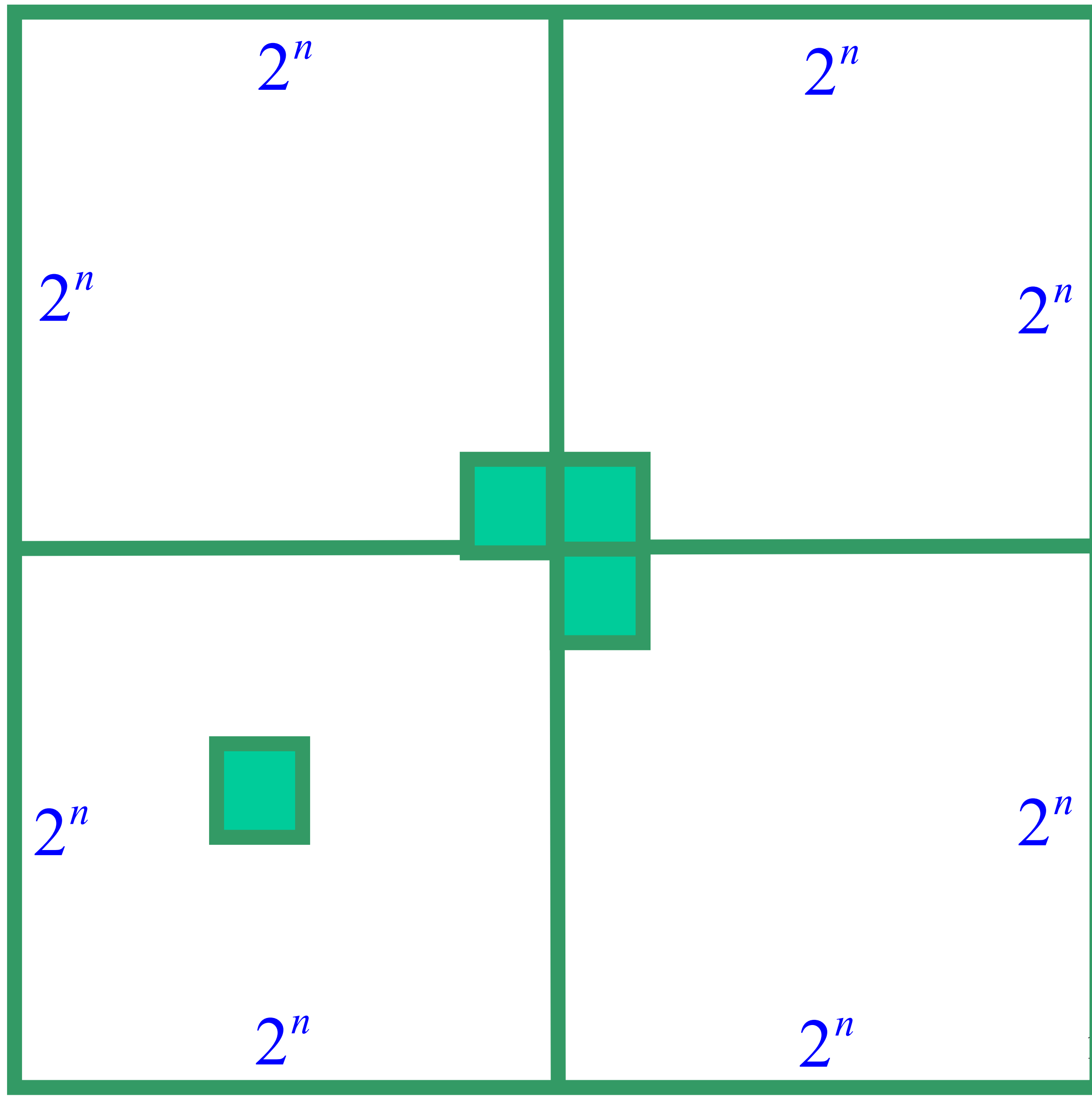
- Show that any $2^n \times 2^n$ chessboard with one square removed can be tiled using L-shaped pieces, each covering three squares.



basis step



inductive step



Template for proof by Induction

- Prove that P(n) is true for all positive integers n
- BASIS STEP: Show that P(1) is true
- INDUCTIVE STEP: Show that P(k)   P(k+1)

Strong Induction

- Prove that P(n) is true for all positive integers n
- BASIS STEP: Show that P(1) is true
- INDUCTIVE STEP: Show that

$$[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$$

Strong Induction example

- Show that if n is an integer greater than 1, n can be written down as a product of primes
- P(n): n can be written down as a product of primes

- Basis Step: P(2) is true, 2=2
- Inductive Step: Assume P(j) is true for all $j \leq k$
  Need to show that P(k+1) is true
- Case 1: k+1 is prime, P(k+1) is true
- Case 2: $k+1 = a \cdot b$

  By induction hypothesis, both a and b can be written as a product of primes.

Proof by contradiction

Prove that $\sqrt{2}$ is irrational

Suppose $\sqrt{2} = \dfrac{m}{n}$ $\quad \gcd(m,n) = 1$

$$2 = \frac{m^2}{n^2}$$

m is even

$$m = 2k$$

$$2 = \frac{4\mathrm{k}^2}{n^2}$$

$$n^2 = 2k^2$$

n is even

Contradiction !