Homework 2: due Nov 27

Problem 1 The game goes like this: Bob holds a number which Alice must find. She can ask Bob questions such as: Is your number bigger than x? To which he replies truthfully. The catch is, she does not know the range within which x lies (it's a positive integer). Here is a strategy: ask if x is bigger than $2, 2^2, 2^3, 2^4$, etc, and do a binary search once the answer is no. This takes $2 \log x + O(1)$ questions. Can Alice do better? (The goal is to minimize the number of questions. The time it takes her to formulate the question is irrelevant.)

Problem 2 Let G be the graph of the n-cube $\{0,1\}^n$: the $N = 2^n$ vertices of G are labeled by n-bit vectors, and two vertices are joined by an edge iff their labels differ by one bit.

1. Prove that G contains a Hamiltonian cycle.

2. Given any two distinct vertices of G, among all the shortest paths joining them, pick one at random uniformly. The $\binom{N}{2}$ paths obtained in this way are called *canonical*. Fix an edge e; give a closed-form expression for the expected number of canonical paths passing through e.

3. Given a subset S of the vertices, let $D(S)$ denote the set of edges with exactly one endpoint in S. Define
$$\varphi = \min_{0<|S|\leq N/2} \frac{|D(S)|}{|S|}.$$
Prove that $\varphi = 1$. Hint: to show that 1 is a lower bound, use the formula you found in 2.

4. Let $M = (M_{ij})$ be the transition matrix of the Markov chain associated with G in the following way: $M_{ii} = 1/2$, $M_{ij} = 1/2n$ if there is an edge between vertices i and j, and $M_{ij} = 0$ otherwise. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of M. As it turns out, $\lambda_2 < 1 - \Omega(\varphi^2/n^2)$ (don't try to prove this!) Explain precisely why this implies that the Markov chain converges towards the uniform distribution in polynomial time, ie, after $n^{O(1)}$ steps, the probability that a random walk is at a given vertex of G is $(1 \pm \varepsilon)/N$, for an arbitrarily small constant $\varepsilon > 0$.