Homework 1: due Oct 30

Problem 1 Let T be a rooted tree, where each internal node has exactly two children (but the tree need not be balanced). Given a node v of T, let $N(v)$ denote the number of leaves in the subtree rooted at v. Prove that

$$\sum_v \min\{\log N(v_1), \log N(v_2)\} = O(n),$$

where the sum runs over all internal nodes v, with v_1 (resp. v_2) denoting the left (resp. right) child of v.

Problem 2 We want to discuss some elementary connections between graphs and linear algebra. Consider an undirected graph G with n vertices and m edges (no loops or multiple edges). Provide an arbitrary orientation to each edge, thus making the graph directed. Now, form the m-by-n matrix D, where each column (resp. row) corresponds to a vertex (resp. edge): The row associated with the directed edge (i, j) has -1 in column i, 1 in column j, and 0 elsewhere.

1. Prove that D has rank $n - c$, where c is the number of connected components.

2. Prove that D is unimodular, ie, each square submatrix of D has determinant $0, 1$, or -1.

3. Define $L = D^T D$. Prove that L is independent of the orientation of G. Let $\mu_0 \leq \mu_2 \leq \ldots \leq \mu_{n-1}$ be the eigenvalues of L. Prove that (i) $\mu_0 = 0$ (what is the corresponding eigenvector?); (ii) $\mu_1 > 0$ if G is connected.

4. Let $E(G)$ denote the logarithm (in base 2) of the number of distinct spanning trees. Using the fact (don't prove it) that the number of spanning trees is equal to $\mu_1 \cdots \mu_{n-1}/n$, show that $E(G) = O(m)$.

Problem 3 A set F of functions from $\{0,1\}^m$ to $\{0,1\}^l$ ($l < m$) is said to form a family of universal hash functions if: given any $x \neq y \in \{0,1\}^m$, a random f chosen uniformly in F is such that

$$\text{Prob}[f(x) = f(y)] \leq 2^{-l} + 2^{-m}.$$

1. Prove that the set F of all linear functions from $\{0,1\}^m$ to $\{0,1\}^l$ forms a family of universal hash functions.

2. Fix a prime p with binary representation of length $m+1$, and let F consist of all functions

$$x \mapsto (ax + b \pmod{p}) \pmod{2^l},$$

for all integers $0 \leq a, b < p$. Prove that F is a family of universal hash functions.