Modeling Transformations

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Overview

- 2D Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
- 3D Transformations
  - Basic 3D transformations
  - Same as 2D
- Transformation Hierarchies
  - Scene graphs
  - Ray casting

2D Modeling Transformations

- Scale
- Rotate
- Translate

Let's look at this in detail…
2D Modeling Transformations

Modeling Coordinates

Scale 0.3, 0.3

2D Modeling Transformations

Modeling Coordinates

Scale 0.3, 0.3

Rotate -90

Transformations can be combined (with simple algebra)

Basic 2D Transformations

• Translation:
  \( x' = x + t_x \)
  \( y' = y + t_y \)

• Scale:
  \( x' = x \times s_x \)
  \( y' = y \times s_y \)

• Shear:
  \( x' = x + h_x y \)
  \( y' = y + h_y x \)

• Rotation:
  \( x' = x \cos \Theta - y \sin \Theta \)
  \( y' = x \sin \Theta + y \cos \Theta \)
### Basic 2D Transformations

- **Translation:**
  - \( x' = x + tx \)
  - \( y' = y + ty \)
- **Scale:**
  - \( x' = x \times sx \)
  - \( y' = y \times sy \)
- **Shear:**
  - \( x' = x + hx'y \)
  - \( y' = y + hy'x \)
- **Rotation:**
  - \( x' = x \cos \Theta - y \sin \Theta \)
  - \( y' = x \sin \Theta + y \cos \Theta \)

### Matrix Representation

- Represent 2D transformation by a matrix
  \[
  \begin{bmatrix}
  a & b \\
  c & d \\
  \end{bmatrix}
  \]
- Multiply matrix by column vector
  \[\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]
  \( x' = ax + by \)
  \( y' = cx + dy \)

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Matrices are a convenient and efficient way to represent a sequence of transformations!
Linear Transformations

- What types of transformations can be represented with a 2x2 matrix?

2D Identity?
\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

2D Scale around (0,0)?
\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \cdot a \\ y \cdot b \end{pmatrix} \]

2D Mirror over Y axis?
\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

2D Mirror over (0,0)?
\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

2D Rotate around (0,0)?
\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

2D Shear?
\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & sy \cdot x \\ sh \cdot y & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

2D Translation?
\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + tx \\ y + ty \end{pmatrix} \]

Only linear 2D transformations can be represented with a 2x2 matrix.

2D Translation represented by a 3x3 matrix
\[ \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \]

- What types of transformations can be represented with a 2x2 matrix?

- Properties of linear transformations:
  - Satisfies: \( \Gamma(s, p_1 + s_2, p_2) = s_1 \Gamma(p_1) + s_2 \Gamma(p_2) \)
  - Origin maps to origin
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved
  - Closed under composition

Point represented with homogeneous coordinates
\[ \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \]
Homogeneous Coordinates
- Add a 3rd coordinate to every 2D point
  - \((x, y, w)\) represents a point at location \((x/w, y/w)\)
  - \((x, y, 0)\) represents a point at infinity
  - \((0, 0, 0)\) is not allowed

Convenient coordinate system to represent many useful transformations

Basic 2D Transformations
- Basic 2D transformations as 3x3 matrices

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
- Translate

\[
\begin{pmatrix}
1 & 0 & t_x \\
0 & 1 & t_y \\
0 & 0 & 1
\end{pmatrix}
\]

- Scale

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

- Rotate

\[
\begin{pmatrix}
\cos\theta & -\sin\theta & 0 \\
\sin\theta & \cos\theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

- Shear

\[
\begin{pmatrix}
1 & s_x & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Affine Transformations
- Affine transformations are combinations of ...
  - Linear transformations, and
  - Translations

\[
\begin{pmatrix}
x' \\
y' \\
w'
\end{pmatrix} = \begin{pmatrix} a & b & c \\
d & e & f \\
0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\
y \\
w \end{pmatrix}
\]

- Properties of affine transformations:
  - Origin does not necessarily map to origin
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved
  - Closed under composition

Projection Transformations
- Projective transformations ...
  - Affine transformations, and
  - Projective warps

\[
\begin{pmatrix}
x' \\
y' \\
w'
\end{pmatrix} = \begin{pmatrix} a & b & c \\
d & e & f \\
g & h & i \end{pmatrix} \begin{pmatrix} x \\
y \\
w \end{pmatrix}
\]

- Properties of projective transformations:
  - Origin does not necessarily map to origin
  - Lines map to lines
  - Parallel lines do not necessarily remain parallel
  - Ratios are not preserved (but “cross-ratios” are)
  - Closed under composition

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  - Ray casting
- 3D Transformations
  - Linear transformations, and
  - Translations

\[
\begin{pmatrix}
x' \\
y' \\
w'
\end{pmatrix} = \begin{pmatrix} a & b & c \\
d & e & f \\
0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\
y \\
w \end{pmatrix}
\]

- Properties of affine transformations:
  - Origin does not necessarily map to origin
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved
  - Closed under composition

Matrix Composition
- Transformations can be combined by matrix multiplication

\[
\begin{pmatrix}
x' \\
y' \\
w'
\end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\
0 & 1 & t_y \\
0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\
\sin\theta & \cos\theta & 0 \\
0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\
y \\
w \end{pmatrix}
\]

- Projection Transformations

\[
\begin{pmatrix}
x' \\
y' \\
w'
\end{pmatrix} = \begin{pmatrix} a & b & c \\
d & e & f \\
g & h & i \end{pmatrix} \begin{pmatrix} x \\
y \\
w \end{pmatrix}
\]

- Properties of projective transformations:
  - Origin does not necessarily map to origin
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Matrix Composition
- Transformations can be combined by matrix multiplication
Matrix Composition

- Matrices are a convenient and efficient way to represent a sequence of transformations
  - General purpose representation
  - Hardware matrix multiply
  - Efficiency with premultiplication
    - Matrix multiplication is associative

\[ p' = (T \cdot (R \cdot (S \cdot p))) \]
\[ p' = (T \cdot R \cdot S) \cdot p \]

Matrix Composition

- Be aware: order of transformations matters
  - Matrix multiplication is not commutative

\[ p' = T \cdot R \cdot S \cdot p \]

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3D Transformations

- Same idea as 2D transformations
  - Homogeneous coordinates: (x,y,z,w)
  - 4x4 transformation matrices

\[
\begin{bmatrix}
  x' \\
  y' \\
  z' \\
  w'
\end{bmatrix} =
\begin{bmatrix}
  a & b & c & d \\
  e & f & g & h \\
  i & j & k & l \\
  m & n & o & p
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  w
\end{bmatrix}
\]

Basic 3D Transformations

- Identity
- Scale
- Translation
- Mirror over X axis
Basic 3D Transformations

Rotate around Z axis:
\[
\begin{pmatrix}
    x' \\
y' \\
z' \\
w'
\end{pmatrix} =
\begin{pmatrix}
    \cos \Theta & -\sin \Theta & 0 & 0 \\
    \sin \Theta & \cos \Theta & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    x \\
y \\
z \\
w
\end{pmatrix}
\]

Rotate around Y axis:
\[
\begin{pmatrix}
    x' \\
y' \\
z' \\
w'
\end{pmatrix} =
\begin{pmatrix}
    \cos \Theta & 0 & -\sin \Theta & 0 \\
    0 & 1 & 0 & 0 \\
    \sin \Theta & 0 & \cos \Theta & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    x \\
y \\
z \\
w
\end{pmatrix}
\]

Rotate around X axis:
\[
\begin{pmatrix}
    x' \\
y' \\
z' \\
w'
\end{pmatrix} =
\begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & \cos \Theta & -\sin \Theta & 0 \\
    0 & \sin \Theta & \cos \Theta & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    x \\
y \\
z \\
w
\end{pmatrix}
\]

Transformation Hierarchies

- Scene may have hierarchy of coordinate systems
  - Each level stores matrix representing transformation from parent’s coordinate system

Robot Arm

Transformation Example 1
- Mike Marr, COS 426, Princeton University, 1995

Transformation Example 2
- An object may appear in a scene multiple times
- Draw same 3D data with different transformations

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Transformation Example 1
- Rose et al. ’96

Transformation Example 2
- Rose et al. ’96

Angel Figures 8.8 & 8.9
Transformation Example 2

Building
Floor 1 Floor 2 Floor 3 Floor 4 Floor 5
Floor Furniture Office 1 Office N
Office Furniture

Bookshelf 1 Desk 1 Desk 2 Chair 1 Chair K
Bookshelf Desk Chair

Definitions

Ray Casting With Hierarchies

• Transform rays, not primitives
  • For each node...
    » Transform ray by inverse of matrix
    » Intersect with primitives
    » Transform hit by matrix

Base
[M1]
Upper Arm
[M2]
Lower Arm
[M3]

Robot Arm

Angel Figures 8.8 & 8.9

Summary

• Coordinate systems
  » World coordinates
  » Modeling coordinates

• Representations of 3D modeling transformations
  » 4x4 Matrices
    » Scale, rotate, translate, shear, projections, etc.
    » Not arbitrary warps

• Composition of 3D transformations
  » Matrix multiplication (order matters)
  » Transformation hierarchies