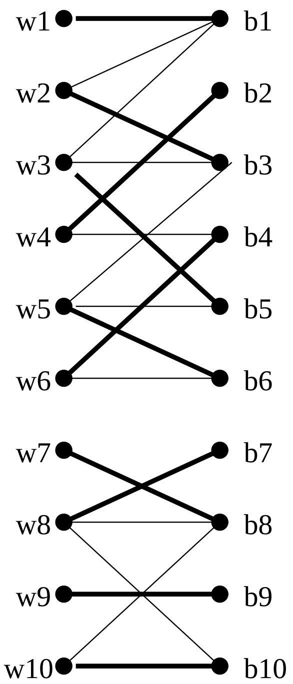
COS 341 Homework 7 Solutions

Section 9.5

Problem 2. Label the board as follows:

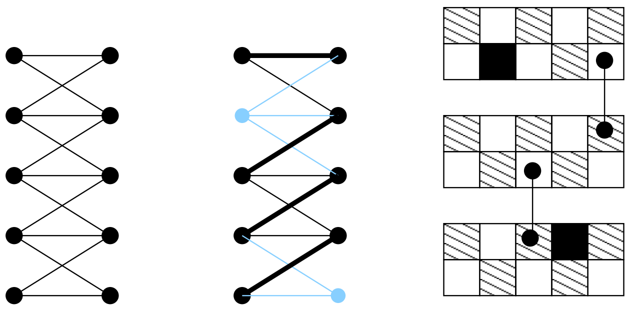
w_1	b_1	w_2	$\times$	$\times$	b_2
$\times$	w_3	b_3	$\times$	b_4	w_4
$\times$	b_5	w_5	b_6	w_6	$\times$
$\times$	$\times$	$\times$	$\times$	$\times$	w_7
$\times$	$\times$	$\times$	b_7	w_8	b_8
b_9	w_9	$\times$	$\times$	b_{10}	w_{10}

The domino graph is then:



The matching of size 10 is indicated in bold.

Problem 6. Consider a chessboard with 2 rows and $n \geq 3$ columns. The domino graph for a $2 \times n$ board is show in figure (a) below for $n = 4$:



(a) (b) (c)

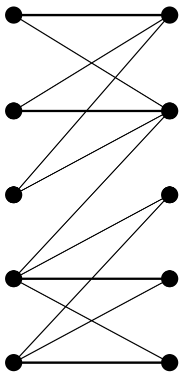
If we remove two squares of opposite colors, we can still find the trivial matching shown in figure (b) above.

Now consider a chessboard with m rows and $n \geq 3$ columns, where m is even and > 2. Remove two squares of different colors. Divide the chessboard into $m/2$ strips of two rows each. See figure (c) above. (In the figure, solid black squares are forbidden). Consider the strips from top of the chessboard to the bottom. If there are no forbidden squares in a strip, we can trivailly tile it. If the two forbidden squares are in the same strip, we can tile that strip by the discussion above. If there is only one forbidden square in a strip, call this strip S and the one below it strip $S+1$. Choose a non-forbidden square of oppositite color along the bottom row of strip S (such a square exists because $n \geq 3$). "Remove" this square and the one adjacent to it in strip $S+1$. These two newly removed squares can be covered by a single domino. Strip S can be tiled by the above discussion because now it contains two forbidden squares. If strip $S+1$ contained the second original forbidden square, now it contains two forbidden squares and can be tiled. If $S+1$ didn't contain the second original forbidden square, now it contains one forbidden square. Repeat the above proceedure. Eventually the strip with the second original forbidden square will be reached and can be tiled.

(This solution based on the one by Alex Halderman.)

An alternate solution ignores graphs and matchings altogether. Divide the board into regions that can trivially be covered and then show how to cover the remaining region containing the forbidden squares.

REMARK Many people said something like "since each vertex has at least two edges adjacent to it, it must have a alternating path." This isn't true in general. A counterexample is:

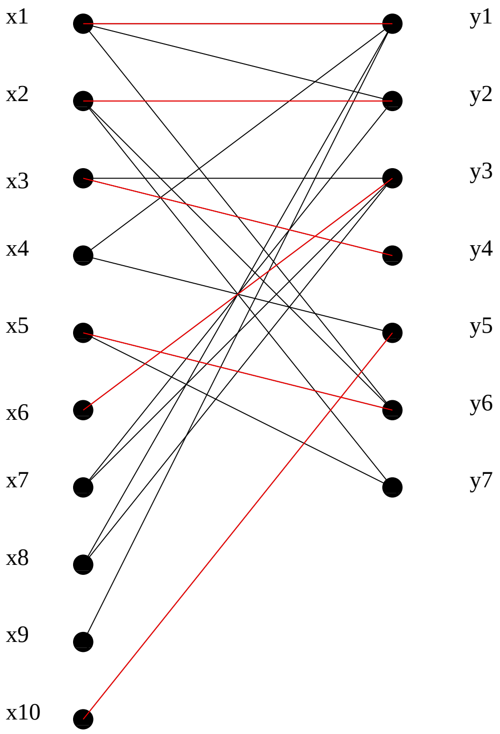


Problem 10. The left graph has a max-matching of size 7. If you started the matching algorithm with the max matching, then no vertices in X are labeled in step 0, and the algorithm stops in step 1 with non-breakthrough. Using Theorem 9.2.3, one possible minimum set cover is $S = X$.

The right graph has a max-matching of size 6. Suppose you start with a size 6 matching, then the algorithm quickly terminates with non-breakthrough. A possible minimum set cover is $S = \{x_1, x_2, x_5, x_6, x_7, y_3\}$.

REMARK The purpose of this exercise was to demonstrate your understanding of the matching algorithm, so a correct solution needed to apply the algorithm in some way.

Problem 11. Set up a bipartite graph in which the applicants make up X, the positions Y, and the qualifications Δ (i.e. if applicant x_i is qualified for position y_j, then there is an edge (x_i, y_j)). Then the problem reduces to finding a maximum matching. The example below has edges in the matching in red. This matching saturates Y thus is a maximum matching.



Special Problem 1.

Part (a) Since Y_j can only be 0 or 1, $\mathbf{E}Y_j = \Pr[Y_j = 1]$. There are $n!$ possible permutations of the n numbers. We need to find out how many in which the jth position is the maximum of the first j elements. We have $\binom{n}{j}$ choices for which numbers to put in the first j spots. These j elements can be in $j!$ permutations, $1/j$ of which will have the maximum in position j. We can order the $n-j$ remaining numbers after the jth position in any of $(n-j)!$ positions. So we have:

$$\Pr[Y_j = 1] = \frac{\frac{1}{j}\binom{n}{j}(n-j)!j!}{n!} = \frac{1}{j}$$

We also have:

$$\mathbf{E}Y = \mathbf{E}\left(\sum_{j=1}^{n} Y_j\right) = \sum_{j=1}^{n} \mathbf{E}Y_j = \sum_{j=1}^{n} \frac{1}{j} = H_n$$

Part (b) An analysis very similar to part (a), just a little more complicated. Again we have $\binom{n}{j}$ ways to pick which numbers will appear in positions $\leq j$. Of these j numbers, we must put the maximum in position j. We then have $\binom{j-1}{k}$ ways to pick which of the remaining $j-1$ numbers will be in positions $\leq k$. Of these first k numbers, we must put the maximum in position k; the remaining $k-1$ numbers can be arranged in any of $(k-1)!$ permutatations. The $j-k-1$ numbers strickly between positions k and j can be arranged in any of $(j-k-1)!$ ways. Finally, we have $(n-j)!$ ways to arrange the numbers at the end after the jth position. Therefore, the number of permutations of $(i_1, \ldots, i_n)$ that have the required property is:

Many people said $\mathbf{E}[Y_jY_k] = \mathbf{E}[Y_j]\mathbf{E}[Y_k]$. This is only true if Y_j and Y_k are independent. If you want to use this fact you must show that Y_j and Y_k are indeed independent.

$$\binom{n}{j}\binom{j-1}{k}(k-1)!(j-k-1)!(n-j)! = \frac{1}{jk}$$

Part (c) We first manipulate Y^2 into a more convienient form:

You needn't have written this out in so much detail; we do so here just for completeness. Though, many people simply explained intuitively where each of the terms comes from — it is better to use a formal argument.

$$\begin{aligned} Y^2 &= \left(\sum_{j=1}^{n} Y_j\right)\left(\sum_{k=1}^{n} Y_k\right) \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} Y_j Y_k \\ &= \sum_{j=1}^{n}\left(Y_j Y_j + \sum_{\substack{k=1 \\ k\neq j}}^{n} Y_i Y_k)\right) \\ &= \sum_{j=1}^{n} Y_j Y_j + \sum_{j=1}^{n}\sum_{\substack{k=1 \\ k\neq j}}^{n} Y_j Y_k \\ &= \sum_{j=1}^{n} Y_j + \sum_{j=1}^{n}\sum_{\substack{k=1 \\ k\neq j}}^{n} Y_j Y_k \end{aligned}$$

where the last equality follows because $Y_j Y_j = Y_j$ for any j since Y_j can only be 0 or 1 variables. Now we take the expectation and by linearity of expectation we get:

$$\mathbf{E}(Y^2) = H_n + \left((H_n)^2 - H_n^2\right)$$

since $(H_n)^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} 1/jk$.

Part (d) Using a useful formula from the handouts, we know that $\mathbf{Var}(X) = \mathbf{E}(X^2) - (\mathbf{E}X)^2$. Plugging in from (a) and (c) above we have

$$\begin{aligned} \mathbf{Var}(Y) &= (H_n + (H_n)^2 - H_n^2) - H_n \\ &= (H_n)^2 - H_n^2 \end{aligned}$$