Homework 6 Solutions

November 21, 2001

Section 7.8 #23, 24, 26, 34, 39 and special problem

7.8 #23

$$\begin{aligned}
&a) \quad \frac{1}{1-cx} \\
&b) \quad \frac{1}{1+x} \\
&c) \quad (1-x)^{\alpha} \\
&d) \quad e^{x} \\
&e) \quad e^{-x}
\end{aligned}$$

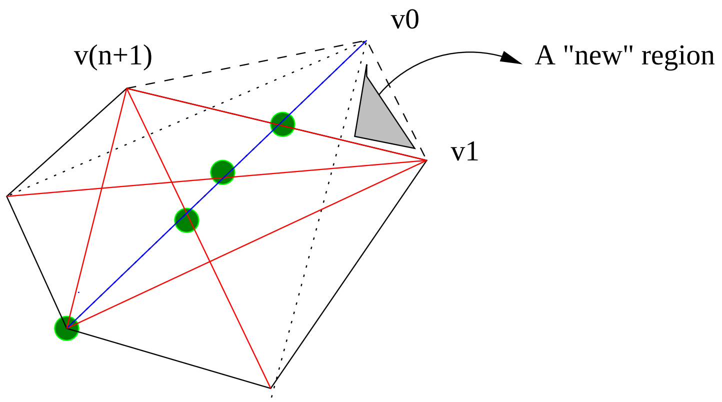
7.8 #24

$$\begin{aligned}
&a) \quad \left(\frac{x}{1-x^2}\right)^4 \\
&b) \quad \left(\frac{1}{1-x^3}\right)^4 \\
&c) \quad (1+x)\left(\frac{1}{1-x}\right)^2 \\
&d) \quad (x+x^3+x^{11})(x^2+x^4+x^5)\left(\frac{1}{1-x}\right)^2 \\
&e) \quad \left(\frac{x^{10}}{1-x}\right)^4
\end{aligned}$$

Note that exponential generating functions are used for n-*permutations*, whereas polynomial generating functions are for n-*combinations*.

7.8 #26 Solve the non-homogeneous recurrence $h_n = 4h_{n-1} + 4^n$, $(n \geq 1)$, and $h_0 = 3$. The characteristic equation $x - 4 = 0$ gives general solution $h_n = c4^n$. For the particular solution, try $h_n = pn4^n = 4p(n-1)4^{n-1} + 4^n \Rightarrow p = 1$. So, $h_n = n4^n + c4^n$. Substituting $h_0 = 3$ gives $c = 3$, and the result is

$$h_n = (3+n)4^n.$$



7.8 #34 Let h_{n-1} denote the number of regions in a convex polygon with $n+1$ vertices. Consider a convex polygon P with $n+2$ vertices. Pick any vertex v_0 of P. The remaining vertices define a convex polygon Q of $n+1$ vertices and has h_{n-1} regions formed by its diagonals. Now add v_0 and the $n+1$ diagonals and sides incident on v_0. Some *new* regions are formed and some *old* regions are split.

First, we note that there are n new regions formed in the triangular region defined by v_0 and its neighbors.

To count the regions obtained by splitting *old* regions, consider adding one diagonal at a time. Each diagonal D_i splits some x_i existing regions into two and the number of regions increases by x_i. (In the figure, the blue diagonal has three intersections with the red segments and hence there are three splits due to the three new segments appearing between the green blobs.) Since the new diagonals do not intersect each other, x_i equals the number of intersections of the new diagonal with the diagonals or side $v_1 v_{n+1}$ of Q.

We shall now count the *total* increase in number of regions, ie, $\sum_{i=1}^{n} x_i$. This number is the total number of intersections of some diagonal D_i and some diagonal (or side $v_1 v_{n+1}$) of Q. To count that, note that whenever we choose three vertices v_i, v_j, v_k $1 \leq i < j < k \leq n+1$ it defines an intersection between segments $v_0 v_j$ and $v_i v_k$ (and every intersection is defined by three vertices). So the number of such intersections = number of ways to choose $v_i, v_j, v_k = \binom{n+1}{3}$.

Thus the increase in number of regions is n from the triangular part and $\binom{n+1}{3}$ from the Q area giving us

$$h_n = h_{n-1} + n + \binom{n+1}{3}$$

Generating Function: Let $g(x) = \sum_{n=0}^{\infty} h_n x^n$. Multiplying the recurrence relation for h_n by x^n, and summing all such equations for $n = 1$ to ∞ we get,

$$g(x) - h_0 = xg(x) + \sum_{n=1}^{\infty} nx^n + \sum_{n=1}^{\infty} \binom{n+1}{3} x^n$$

But we have $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ and $\sum_{n=1}^{\infty} = \frac{x^2}{(1-x)^4}$. With $h_0 = 0$, after some rearrangement, we get

$$g(x) = \frac{x}{(1-x)^3} + \frac{x^2}{(1-x)^5}$$

To find h_n we find the coefficient of x^n in $g(x)$. We can use equation (7.47) in the textbook:

$$\begin{aligned} g(x) &= \frac{x}{(1-x)^3} + \frac{x^2}{(1-x)^5} \\ &= \sum_{n=0}^{\infty} \binom{n+2}{2} x^{n+1} + \sum_{n=0}^{\infty} \binom{n+4}{4} x^{n+2} \\ &= \sum_{n=0}^{\infty} \binom{n+1}{2} x^n + \sum_{n=0}^{\infty} \binom{n+2}{4} x^n \end{aligned}$$

Hence,

$$h_n = \binom{n+1}{2} + \binom{n+2}{4}$$

(This solution by Manoj Prabhakaran.)

7.8 #39 The number of ways to be colored is

$$\begin{aligned} &(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots)^2 (1 + x + \frac{x^2}{2!} + \cdots)^2 \\ &= \left(\frac{e^x + e^{-x}}{2}\right)^2 (e^x)^2 \\ &= \frac{e^{4x}}{4} + \frac{e^{2x}}{2} + \frac{1}{4}. \end{aligned}$$

The coefficient of $x^n/n!$ in this generating function is $h_n = 4^{n-1} + 2^{n-1}$.

Special problem $f(x) = (1 + x^2 + x^4 + ...)(\sum_{n=0}^{\infty} \binom{1/3}{i} x^i)$. The coefficients of x^4 and x^5 are

$$a_4 = \binom{1/3}{0} + \binom{1/3}{2} + \binom{1/3}{4} = \frac{206}{243}.$$

$$a_5 = \binom{1/3}{1} + \binom{1/3}{3} + \binom{1/3}{5} = \frac{310}{729}.$$