# Homework 5 Solutions

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Section 7.8 #2,12,22 and special problems

**7.8 #2** We know that the $n$th Fibonacci number is

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for $n \geq 0$. The right and left terms sum to a integer. Since $1/\sqrt{5} < 1/2$ and $|1-\sqrt{5}|/2 < 1$, the left term is always less than 1/2, so $(1+\sqrt{5})/2$ is the integer closest to $f_n$.

**7.8 #12** We solve the linear homogeneous recurrence relation

$$h_n = 8h_{n-1} - 16h_{n-2} \quad (n \geq 2, h_0 = -1, h_1 = 0).$$

The characteristic equation $x^2 - 8x + 16 = 0$ has roots $x = 4$. The general solution is $c_1 4^n + c_2 n 4^n$. Substituting the initial cases gives us $c_1 = -1, c_2 = 1$, so the solution is $h_n = -4^n + n4^n$.

**7.8 #22** Now we solve the nonhomogeneous recurrence relation

$$h_n = 4h_{n-1} - 4h_{n-2} + 3n + 1 \quad (n \geq 2, h_0 = 1, h_1 = 2).$$

First, the homogeneous part. Characteristic equation $x^2 - 4x + 4 = 0$ has roots $x = 2$, for general equation $h_n = c_1 2^n + c_2 n 2^n$. Then the nonhomogeneous part: $h_n = rn+s = 4(r(n-1)+2) - 4(r(n-2)+2) + 3n+1 = 4r+3n+1$. This gives $r = 3, s = 13$. Combining the two, we have $h_n = c_1 2^n + c_2 n 2^n + 3n + 13$. Substituting with $h_0$ and $h_1$ gives $c_1 = -12, c_2 = 5$, for result $h_n = -12(2^n) + 5n2^n + 3n + 13$.

**Special Problem 1** The probability that a random $n$ is a winner is

$$p = \frac{m = \#\text{ integers } 1 \leq n \leq t^5 \text{ that satisfy } \lfloor n^{1/5} \rfloor \text{ divides } n}{N = t^5}.$$

Let $A_j$ be the set of integers $n$ that satisfy $\lfloor n^{1/5} \rfloor = j$. The numerator $m = \sum_{1 \leq j \leq t-1} |A_j| + 1$. (Last term is for $t^5$.) $A_j = \lceil \frac{(j+1)^5 - 1 - j^5 + 1}{j} \rceil = \lceil 5j^3 + 10j^2 + 10j + 5 + 1/j \rceil = 5j^3 + 10j^2 + 10j + 6$.

Recall

$$\sum_{1\le n\le x} n^3 = \frac{x^2(x+1)^2}{4}, \qquad \sum_{1\le n\le x} n^2 = \frac{x(x+1)(2x+1)}{6}, \qquad \sum_{1\le n\le x} n = \frac{x(x+1)}{2}.$$

So

$$\begin{aligned} m &= 1 + \sum_{1\le j\le t-1} (5j^3 + 10j^2 + 10j + 6) \\ &= 1 + 6\frac{(t-1)^2t^2}{4} + 10\frac{(t-1)(t)(2t-1)}{6} + 10\frac{(t-1)t}{2} + 6(t-1). \end{aligned}$$

Expanding and simplifying gives

$$p = -\frac{5}{t^5} + \frac{8}{3t^4} + \frac{5}{4t^3} + \frac{5}{6t^2} + \frac{5}{4t}.$$

**Special Problem 2**

a. $\Omega = (U, p)$. Represent each team by a 0 or 1.
$U = \{x_1x_2...x_n \mid n \le 7, x_i \in \{0,1\}, \sum x_i = 4 \text{ or } \sum x_i = n-4\}$.
$p(x_1 \dots x_n) = (1/2)^n$.

Many people assumed that the probability for each $x \in U$ was the same. In fact, since the strings in $U$ are not the same length, the probability depends on the length of the string.

b. Pr[go into 7th game] = sum of Pr[outcome] for each outcome that requires a 7th game. Pr[outcome] = $(1/2)^7$. The number of such outcomes is the number of permutations of three 0's and three 1's followed by a 1 or 0, or $\frac{6!}{3!3!} \cdot 2 = 40$. So the solution is $40/2^7 = 5/16$.

c.

$$\begin{aligned} e[X] &= \sum_{n=\#\text{ games}} n \cdot \Pr[n \text{ game outcome}] \\ &= \sum_{4\le i\le 7} i \cdot \left(\frac{1}{2}\right)^i \binom{i-1}{3} \cdot 2 \\ &= 2\left[\frac{4}{2^4}\binom{3}{3} + \frac{5}{2^5}\binom{4}{3} + \frac{6}{2^6}\binom{5}{3} + \frac{7}{2^7}\binom{6}{3}\right] \\ &= 372/64 = 5.8125 \end{aligned}$$

Following above, $Var[X] = 263/154 = 1.027$.

d Similar to [b]: The probability of a $n$-game outcome is $(1/2)^n$. The number of such games is the permutation of $(n-1)/2$ wins for each of the two teams, followed by the winning game, or $\binom{n-1}{(n-1)/2}$. Result is $(1/2)^n\binom{n-1}{(n-1)/2}$.