COS 341 Homework 4 Solutions

Section 5.8

Problem 18. We evaluate $E = \sum_{i=0}^{n}(-1)^i \frac{1}{i+1}\binom{n}{i}$. We have first that $\frac{1}{i+1}\binom{n}{i} = \frac{1}{n+1}\binom{n+1}{i+1}$. So we evaluate $E = \frac{1}{n+1}\sum_{i=0}^{n}(-1)^i\binom{n+1}{i+1}$. In section 5.3 of the textbook we can find the fact $\sum_{i=-1}^{n}(-1)^i\binom{n+1}{i+1} = 0$. Hence we have $E = \frac{1}{n+1}\left(\sum_{i=0}^{n}(-1)^i\binom{n+1}{i+1} - (-1)\binom{n+1}{0}\right)$, and therefore $E = \frac{1}{n+1}$.

Problem 20. We expand the right hand side of $m^3 = a\binom{m}{3} + b\binom{m}{2} + c\binom{m}{1}$ and equate the coefficients of the powers of m. We get:

$$a\frac{m(m-1)(m-2)}{6} + b\frac{m(m-1)}{2} + cm = \frac{a}{6}(m^3 - 3m^2 + 2m) + \frac{b}{2}(m^2 - m) + cm$$

Since the coefficient of m^3 must be 1, we have $a = 6$. We know the coefficient of m^2 is 0, so we have $(b/2 - 3) = 0$, so $b = 6$. Finally, we get $c = 1$.

Special Problem 1. Begin with the binomial expansion of $(x+1)^n$:

$$(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^k.$$

Take the derivative to put a k into the sum:

$$n(x+1)^{n-1} = \sum_{k=0}^{n}k\binom{n}{k}x^{k-1}.$$

Multiply by x to fix the exponent:

$$nx(x+1)^{n-1} = \sum_{k=0}^{n}k\binom{n}{k}x^k.$$

Finally, let $x = 7$ and we get:

$$7 \cdot n \cdot 8^{n-1} = \sum_{k=1}^{n}k7^k\binom{n}{k}$$

which is the desired result.

Special Problem 2. Begin with equation (5.11) from the book, and multiply both sides by $\binom{2n}{n}$:

Many people tried to square both sides of equation (5.11). This is much harder.

$$\begin{aligned}\binom{2n}{n}^2 &= \binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2 \\ &= \frac{(2n)!}{n!n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{n!}{k!(n-k)!} \\ &= \sum_{k=0}^{n}\frac{(2n)!}{(k!)^2(n-k)^2}\end{aligned}$$

Special Problem 3. Consider the following functions:

This solution is simpler than it looks. Also, there are other (possibly shorter) solutions. This one is given here because it makes the cancelling obvious.

$$\begin{aligned}(1+u)^n &= \sum_{k=0}^{n}\binom{n}{k}u^k \\ (1+u^2)^n &= \sum_{k=0}^{n}\binom{n}{k}u^{2k} \\ (1+1)^n &= \sum_{k=0}^{n}\binom{n}{k}\end{aligned}$$

We then expand each sum as follows:

$$\begin{aligned}(1+u)^n &= \sum_{\substack{k=0\\ k \bmod 3=0}}^{n}\binom{n}{k}u^k + \sum_{\substack{k=0\\ k \bmod 3=1}}^{n}\binom{n}{k}u^k + \sum_{\substack{k=0\\ k \bmod 3=2}}^{n}\binom{n}{k}u^k \\ (1+u^2)^n &= \sum_{\substack{k=0\\ k \bmod 3=0}}^{n}\binom{n}{k}u^{2k} + \sum_{\substack{k=0\\ k \bmod 3=1}}^{n}\binom{n}{k}u^{2k} + \sum_{\substack{k=0\\ k \bmod 3=2}}^{n}\binom{n}{k}u^{2k} \\ (1+1)^n &= \sum_{\substack{k=0\\ k \bmod 3=0}}^{n}\binom{n}{k} + \sum_{\substack{k=0\\ k \bmod 3=1}}^{n}\binom{n}{k} + \sum_{\substack{k=0\\ k \bmod 3=2}}^{n}\binom{n}{k}\end{aligned}$$

Now, note that $u^{3j} = 1, u^{3j+1} = u, u^{3j+2} = u^2$ for any integer j and that $u^4 = u$. Using these identities, we rewrite the above:

$$\begin{aligned}(1+u)^n &= \sum_{\substack{k=0\\ k \bmod 3=0}}^{n}\binom{n}{k} + \sum_{\substack{k=0\\ k \bmod 3=1}}^{n}\binom{n}{k}u + \sum_{\substack{k=0\\ k \bmod 3=2}}^{n}\binom{n}{k}u^2 \\ (1+u^2)^n &= \sum_{\substack{k=0\\ k \bmod 3=0}}^{n}\binom{n}{k} + \sum_{\substack{k=0\\ k \bmod 3=1}}^{n}\binom{n}{k}u^2 + \sum_{\substack{k=0\\ k \bmod 3=2}}^{n}\binom{n}{k}u \\ (1+1)^n &= \sum_{\substack{k=0\\ k \bmod 3=0}}^{n}\binom{n}{k} + \sum_{\substack{k=0\\ k \bmod 3=1}}^{n}\binom{n}{k} + \sum_{\substack{k=0\\ k \bmod 3=2}}^{n}\binom{n}{k}\end{aligned}$$

Now sum these three equations vertically. Because $1 + u + u^2 = 0$, all columns cancel except the first. This first column is 3 times the desired sum. Therefore,

$$g(n) = \frac{(1+u)^n + (1+u^2)^n + 2^n}{3}$$

Special Problem 4. We know there are $\binom{2n}{n}$ different paying patterns. Let D be the set of paying patterns in which the cashier goes into debt. Based on the hint, we conjecture that $|D| = \binom{2n}{n+1}$. We prove this below by constructing a one-to-one and onto mapping from D onto the set of strings of length $2n$ with exactly $n + 1$ ones.

It was not necessary to go into this much detail on the homework (though to do so would have been okay).

We construct the mapping f as follows. Let $a_1a_2 \ldots a_k a_{k+1} \ldots a_{2n}$ be a paying pattern in D where k is the first position that the cashier goes into debt. Take $f(a_1a_2 \ldots a_k a_{k+1} \ldots a_{2n}) = \overline{a_1 a_2 \ldots a_k} a_{k+1} \ldots a_{2n}$, where $\overline{x}$ denotes bitwise compliment.

Since the cashier goes into debt for the first time at a_k, $a_1 a_2 \ldots a_k$ will contain exactly one more zero than one. Thus, $\overline{a_1 a_2 \ldots a_k}$ contains exactly one more one than zero. So $\overline{a_1 a_2 \ldots a_k} a_{k+1} \ldots a_{2n}$ contains $n+1$ ones and $n-1$ zeros, as desired.

We first show that f is onto: that for every string y that contains $n+1$ ones there is some paying pattern x in D such that $f(x) = y$. Given y construct x as follows: since y has $n+1$ ones, it can be written in the form $a1b$ where there are an equal number of zeros and ones in a and a is as short as possible. Let $x = \overline{a}0b$; then $x \in D$ because the cashier will go into debt at the 0 after a, and $f(x) = y$.

Most people forgot to show that their mapping was onto. This is a very important part of the proof, since you are using the size of the range of your mapping to determine the size of your domain.

Now, we need to show that f is one-to-one. Let $x \neq y$ be two paying patterns in D. We show that $f(x) \neq f(y)$. Let k_x, k_y be the first positions at which the cashier goes into debt in x and y respectively. If $k_x = k_y$, then clearly $f(x) \neq f(y)$. Otherwise, assume without loss of generality, that $k_x < k_y$. Let i be the first position at which x and y differ. We know $i < k_x$, otherwise k_x would equal k_y. But then it is clear that $f(x) \neq f(y)$.