COS 341 Homework 2 Solutions

Section 3.6

Problem 10. Sets of 3 numbers out of 1..20: 20! / (3! 17!) = 1140.

Sets of 3 numbers in which at least 2 are consecutive: 19 consecutive pairs $\times$ 18 ways to select the third number $-$ 18, because we counted sets of 3 consecutive numbers twice.

$1140 - 324 = 816$.

Problem 19. # of circular permutations of 10 objects = 9!.

To get the # of circular permutations of 10 objects in which 2 specific objects are opposite, fix the two opposite objects, then there are 8! permutations of the remaining.

Total $= 9! - 8!$

Problem 23. We are looking for circular permutations. There are 4! ways to assign the men circularly, then 5! ways to assign the women so each woman is between two men. Then there are 10 places to insert the dog.

Total $= 4! \times 5! \times 10$

Problem 26. There are 12 total objects. We can build up groups of 10 in the following way:

$$\begin{aligned}
\{3a, 4b, 3c\} &\Rightarrow 10!/3!4!3! = 4200 \\
\{3a, 3b, 4c\} &\Rightarrow 4200 \\
\{3a, 2b, 5c\} &\Rightarrow 10!/3!2!5! = 2520 \\
\{2a, 4b, 4c\} &\Rightarrow 3150 \\
\{2a, 3b, 5c\} &\Rightarrow 2520 \\
\{1a, 4b, 5c\} &\Rightarrow 1260
\end{aligned}$$

Sum of above is 17850

Problem 28. 3-combinations:

{2 a, 1 b}
{2 a, 1 c}
{1 a, 1 b, 1 c}
{1 a, 2 c}
{1 b, 2 c}
{3 c}

4-combinations:

{2 a, 1 b, 1 c}
{2 a, 2 c}
{1 a, 1 b, 2 c}
{1 a, 3 c}
{1 b, 3 c}

Special Problem

Part (a) Failure if:

- both 1 and 2 are among the first k numbers:

$$\begin{array}{c} 1\ 2\ 3\ 4 \\ 1\ 2\ 4\ 3 \\ 2\ 1\ 3\ 4 \\ 2\ 1\ 4\ 3 \end{array}$$

- if 1 is among the first k numbers and 2 is not the nth number

$$\begin{array}{c} 1\ 3\ 2\ 4 \\ 3\ 1\ 2\ 4 \\ 1\ 4\ 2\ 3 \\ 3\ 1\ 2\ 3 \end{array}$$

All other permutations lead to success. So $q_{4,2} = 16/24 = 2/3$.

Part (b)

Notice that Pr[success] $= Pr$[strategy returns 1] $+ Pr$[strategy returns 2].

Pr[strategy returns 1] is exactly the $p_{n,k}$ from class. We can compute Pr[strategy returns 2] similarly.

First, if 1 is among the first k numbers then 2 has to be the nth number. We get a permutation satisfying this condition with probability $k \times (n-2)!/n!$, since there are k ways to place the 1 and $(n-2)!$ ways to permute numbers $3, \ldots, n$.

If 1 is not among the first k numbers, then it has to come after 2. Note that both of them fall within the last $n-k$ numbers. Following the strategy and notation used in class: Let T_j be the set of all permutations where $i_j = 2$, and 1 comes after it, i.e. 1 is among $i_{j+1}, ..., i_n$. Then the total number of permutations that let us select 2 is the sum of T_j for $k < j < n$ (instead of $j \leq n$).

We describe each permutation as (I_1, I_2), where I_1 is a $(n-j)$-permutation of $\{1, 3, 4, ..., n\}$ to fix the values of $i_{j+1}, \ldots, i_n$, and I_2 is a $(j-1)$-permutation $i_1, \ldots, i_{j-1}$. $|T_j| = |I_1| \times |I_2|$. (Remember this is # permutations, not the probability.) There are $n-j$ positions after i_j. Then there are $n-j$ ways to assign the 1, and $P(n-2, n-j-1) = (n-2)!/(j-1)!$ to assign the other $n-j-1$ positions, for a total of $(n-j)(n-2)!/(j-1)!$ permutations for $|I_1|$.

To determine $|I_2|$, we ask, how many permutations of $j-1$ elements have their minimum located in the first k positions? This is the same question from the handout, which shows that $|I_2| = |W| = k(j-2)!$.

Putting this together, we get

$$\begin{aligned} q_{n,k} &= p_{n,k} + k(n-2)!/n! + \sum_{k<j<n} (n-j)(n-2)!k(j-2)!/(j-1)!n! \\ &= p_{n,k} + k/(n(n-1)) + \sum_{k<j<n} k(n-j)/(n(n-1)(j-1)) \end{aligned}$$

There are several equivalent forms.

For $q_{4,2}$, this gives $5/12 + 1/6 + 1/12 = 2/3$, as desired.