COS 341, January 2002

Solutions to Final Exam

Problem 1 (a) Let $n > 0$ be any even integer. By grouping every two consecutive terms in the summation, we have

$$\begin{aligned}
f(n) &= \sum_{1 \le j \le n/2} ((2j)^2 - (2j-1)^2) \\
&= \sum_{1 \le j \le n/2} (4j - 1) \\
&= 4 \sum_{1 \le j \le n/2} j - \sum_{1 \le j \le n/2} 1 \\
&= 4 \frac{1}{2} \frac{n}{2} (\frac{n}{2} + 1) - \frac{n}{2} \\
&= \frac{n(n+1)}{2}. \qquad (1)
\end{aligned}$$

Let $n > 0$ be any odd integer. We will prove

$$f(n) = -\frac{n(n+1)}{2}. \qquad (2)$$

Equation (2) is obviously true for $n = 1$. If $n > 1$, by separating the last term in the summation, we have

$$f(n) = f(n-1) - n^2.$$

Since $n-1 > 0$ is even, we have from (1) that $f(n-1) = (n-1)n/2$. Thus, $f(n) = (n-1)n/2 - n^2 = -n(n+1)/2$. This proves (2).

From (1), (2) we conclude that, for all integers $n > 0$,

$$f(n) = (-1)^n \frac{n(n+1)}{2}.$$

(b) Note that

$$\begin{aligned}
\frac{1}{2}(e^x + e^{-x}) &= \frac{1}{2}(\sum_{k \ge 0} \frac{x^k}{k!} + \sum_{k \ge 0} (-1)^k \frac{x^k}{k!}) \\
&= \sum_{k=0,2,4,6,8,\ldots} \frac{x^k}{k!} \\
&= \sum_{j \ge 0} \frac{x^{2j}}{(2j)!}.
\end{aligned}$$

It follows that

$$\begin{aligned}
g(n) &= \sum_{j \ge 0} \frac{(n^2)^{2j}}{(2j)!} \\
&= \frac{1}{2}(e^{n^2} + e^{-n^2}).
\end{aligned}$$

Problem 2 (a) The number of 1's in β is $\sum_{1\le i\le k} b_k$, and the number of 0's in β is $\sum_{1\le i\le k}(1-b_k)$. It follows that

$$\delta(\beta) = |\sum_{1\le i\le k} b_k - \sum_{1\le i\le k}(1-b_k)| = |-k + 2\sum_{1\le i\le k} b_k|.$$

(b) There are exactly two 1-bit strings that are balanced (ie, 0 and 1). There are exactly two 2-bit strings that are balanced (ie, 01 and 10). Thus,

$$c_1 = c_2 = 2. \tag{3}$$

Let $n > 2$, the only n-bit balanced strings are of the form 01α or 10α where α can be any $(n-2)$-bit balanced strings. This gives

$$c_n = 2c_{n-2} \quad \text{for all} \quad n > 2. \tag{4}$$

Iterating (4) and applying (3) we get $c_n = 2^{n/2}$ if n is even, and $c_n = 2^{(n+1)/2}$ if n is odd. It follows that, for all integer $n > 0$,

$$c_n = 2^{\lceil n/2\rceil}.$$

(c) Let D_n be the set of all n-bit almost-balanced strings. Then by definition, $d_n = |D_n|$. For any k-bit string β, where $1 \le k \le n$, let $D_{n,\beta}$ be the set of strings $\alpha \in D_n$ that have β as a prefix, ie, those $\alpha \in D_n$ of the form $\beta\gamma$ for some binary string γ.

Lemma 1 For all $n \ge 1$, $|D_{n,0}| = |D_{n,1}| = d_n/2$.

Proof of Lemma 1 By symmetry between the letters 0 and 1, $|D_{n,0}| = |D_{n,1}|$. Observe also $|D_n| = |D_{n,0}| + |D_{n,1}|$, since D_n is the disjoint union of $D_{n,0}$ and $D_{n,1}$. This proves $d_n = |D_n| = 2|D_{n,0}| = 2|D_{n,1}|$. This proves Lemma 1.

Lemma 2 For all $n \ge 3$, $|D_{n,01}| = |D_{n,10}| = d_{n-2}$, and $|D_{n,00}| = |D_{n,11}| = d_{n-2}/2$.

Proof of Lemma 2 Note that $D_{n,01}$ consists of exactly those n-bit strings of the form 01γ such that $\gamma \in D_{n-2}$. Thus, $|D_{n,01}| = |D_{n-2}| = d_{n-2}$. By symmetry, $|D_{n,10}| = d_{n-2}$. This proves half of the equations we need to establish.

To prove the other half, note that $D_{n,001}$ consists of exactly those n-bit strings of the form 001γ such that $0\gamma \in D_{n-2}$. This gives $|D_{n,001}| = |D_{n-2,0}|$. Using Lemma 1, we get

$$|D_{n,001}| = d_{n-2}/2. \tag{5}$$

Since $D_{n,00} = D_{n,001} \cup D_{000}$ and $D_{n,000}$ is empty (no string starting with 000 can be almost-balanced), we have $|D_{n,00}| = |D_{n,001}|$. Using (5) we then obtain $|D_{n,00}| = d_{n-2}/2$. By symmetry we have also $|D_{n,11}| = d_{n-2}/2$. This completes the proof of Lemma 2.

Lemma 3 $d_1 = 2, d_2 = 4$, and for all $n \ge 3$,

$$d_n = 3d_{n-2}.$$

Proof of Lemma 3 Note that $D_1 = \{0,1\}, D_2 = \{00, 01, 10, 11\}$, and so $d_1 = 2, d_2 = 4$. It remains to prove that, for all $n \ge 3$, $d_n = 3d_{n-2}$.

Let $n \geq 3$. Clearly, D_n is the disjoint union $D_{n,01} \cup D_{n,10} \cup D_{n,00} \cup D_{n,11}$. Thus,

$$d_n = |D_n| = |D_{n,01}| + |D_{n,10}| + |D_{n,00}| + |D_{n,11}|. \tag{6}$$

Lemma 2 and (6) imply $d_n = 3d_{n-2}$. This completes the proof of Lemma 3.

As in part (a), by iterating the recurrence relation given in Lemma 3, we obtain, for all $n > 0$,

$$\begin{aligned} d_n &= 2 \cdot 3^{(n-1)/2} \quad \text{if } n \text{ is odd,} \\ d_n &= 4 \cdot 3^{(n-2)/2} \quad \text{if } n \text{ is even.} \end{aligned}$$

Problem 3 Let

$$\phi(k, m) = (k-1)^m + (-1)^m (k-1). \tag{7}$$

From Homework Set 9 (Exercise 13.6-13), we know that for the m-cycle graph C_m, we have

$$P_{C_m}(k) = \phi(k, m). \tag{8}$$

(a) First some preliminaries. Let v be any fixed vertex of any m-cycle C_m. Suppose we want to color the vertices with colors $1, 2, \cdots, k$. Let i be any fixed color. Let $\psi_i(k, m)$ be the number of k-vertex-colorings of C_m that assign color i to v. By symmetry this quantity is independent of i. Using (8) we have $\phi(k, m) = P_{C_m}(k) = \sum_{1 \leq i \leq k} \psi_i(k, m)$, and hence

$$\psi_i(k, m) = \frac{\phi(k, m)}{k}. \tag{9}$$

To evaluate P_{F_m}, observe that every k-vertex coloring of F_m can be obtained uniquely by first assigning some color i to this common vertex, and then independently finishes coloring vertices on each of F_m's two m-cycles. This, with the help of (9), leads to the answer:

$$\begin{aligned} P_{F_m}(k) &= \sum_{1 \leq i \leq k} (\psi_i(k, m))^2 \\ &= \sum_{1 \leq i \leq k} (\frac{\phi(k, m)}{k})^2 \\ &= \frac{(\phi(k, m))^2}{k}. \end{aligned}$$

(b) We now evaluate $P_{G_n}(k)$. By applying equation (13.1) in the textbook to edge e, we have

$$P_{G_n}(k) = P_{G'_n}(k) - P_{G''_n}(k), \tag{10}$$

where G'_n is obtained from G_n by deleting the edge $e = \{x_0, x_{n/2}\}$, and G''_n is obtained from G_n by contracting the vertices $x_0, x_{n/2}$. Clearly, G'_n is an n-cycle, and G''_n is an $F_{n/2}$. Hence by (8) $P_{G'_n}(k) = \phi(k, n)$, and by the result from part (a), $P_{G''_n}(k) = P_{F_{n/2}}(k) = (\phi(k, n/2))^2/k$. From (10) we have then

$$P_{G_n}(k) = \phi(k, n) - \frac{(\phi(k, n/2))^2}{k}.$$

(c) For any integer $m > 1$, let L_m be the graph which consists of a simple path on m vertices. (That is, L_m is obtained from C_m by deleting one edge.) Let v_1, v_m be the two vertices of degree 1. Suppose we want to color the vertices of L_m with colors $1, 2, \cdots, k$. For any fixed colors i, j, let $\lambda_{i,j}(k,m)$ be the number of k-vertex-colorings of L_m that assign color i to v_1 and color j to v_m. Clearly, for each i

$$\lambda_{i,i}(k,m) = \psi_i(k, m-1),$$

and so by (9)

$$\lambda_{i,i}(k,m) = \frac{\phi(k, m-1)}{k}. \tag{11}$$

We also assert that for any $i \neq j$

$$\lambda_{i,j}(k,m) = \frac{\phi(k,m)}{k(k-1)}. \tag{12}$$

The proof is similar to the derivation of (9). By symmetry, $\lambda_{i,j}(k,m)$ is independent of the values of i, j as long as $i \neq j$. As $P_{C_m}(k) = \sum_{i,j,i\neq j} \lambda_{i,j}(k,m)$, we must have for any $i \neq j$, $\lambda_{i,j}(k,m) = P_{C_m}(k)/k(k-1)$. Equation (12) then follows from (8).

We now evaluate $P_{H_n}(k)$. Let $A_{i,j}(k,n)$ be the number of k-vertex-colorings of H_n that assign color i to x_0 and color j to $x_{n/2}$. Clearly,

$$P_{H_n}(k) = \sum_{1\leq i\leq k} \sum_{1\leq j\leq k} A_{i,j}(k,n). \tag{13}$$

One can think of the graph H_n as three disjoint chains (of lengths $n/2, n/2, n+1$, respectively) sharing the same two endpoints $x_0, x_{n/2}$. Let i, j be any fixed colors. Any k-vertex coloring of H_n, with x_0 colored i and $x_{n/2}$ colored j, can be uniquely obtained by independently completing the coloring of each of the three disjoint chains. Thus, for each $1 \leq i \leq k$,

$$A_{i,i}(k,n) = (\lambda_{i,i}(k, 1+\frac{n}{2}))^2 \lambda_{i,i}(k, n+2); \tag{14}$$

and for every pair i, j with $i \neq j$,

$$A_{i,j}(k,n) = (\lambda_{i,j}(k, 1+\frac{n}{2}))^2 \lambda_{i,j}(k, n+2). \tag{15}$$

From (11)-(15), we obtain the answer to part (c)

$$\begin{aligned} P_{H_n}(k) &= \sum_i A_{i,i}(k,n) + \sum_{i,j,i\neq j} A_{i,j}(k,n) \\ &= k\left(\frac{\phi(k,\frac{n}{2})}{k}\right)^2 \frac{\phi(k,n+1)}{k} + k(k-1)\left(\frac{\phi(k,1+\frac{n}{2})}{k(k-1)}\right)^2 \frac{\phi(k,n+2)}{k(k-1)}. \end{aligned}$$

(Recall that from (7) that by definition $\phi(k,m) = (k-1)^m + (-1)^m(k-1)$.)