Satisfiability Encodings

Introduction to
Artificial Intelligence

COS302

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Administration

Questions?

Clarify 3-CNF conversion?

RISC AI

- Much design and implementation work has gone into creating SAT solvers, like chaff.
- By “compiling” problems to SAT, we take advantage of this effort.

3-Coloring as SAT

- Arguably fastest way to solve coloring problems.
- We are given a graph with a set of nodes N and edges E.
- One encoding:
  - For each node i, three Boolean variables Ri, Bi, Gi representing color.

Coloring Clauses

No conflicting colors:

- **All (i, j): (~Ri+~Rj)(~Bi+~Bj)(~Gi+~Gj)**

No uncolored nodes:

- **All i: (Ri+Bi+Gi)**

No multicolored nodes:

- **All i: (~Ri+~Bi)(~Ri+~Gi)(~Bi+~Gi)**

General CSPs

- CSP: V, D, C
- For all v in V and d in D, Boolean variable A(v,d) that says we are assigning variable v value d.
- For each entry e in each legal assignment list, a variable L(e) that says we've chosen the corresponding legal assignment.

CSP Clauses

Must assign each variable:

- All v: (A(v,d1)+A(v,d2)+...)

Can't assign multiple values:

- All v, d1, d2: (~A(v,d1)+~A(v,d2))

Must pick a legal assignment:

- All constraints: (L(e1)+L(e2)+...)

Apply chosen assignment:

- All e, v: (~L(e)+A(v,d(e,v))

Battleships

Fleet hiding in a 10x10 grid (section of ocean).

1  Battleship

2  Cruisers

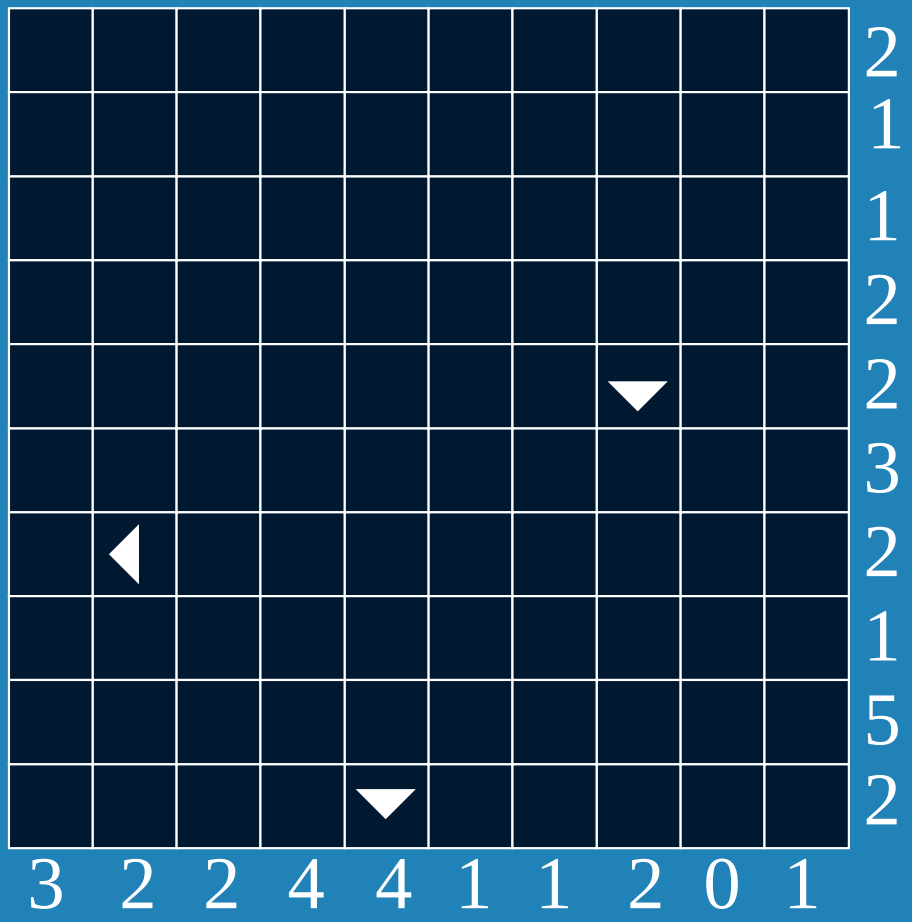
3  Destroyers

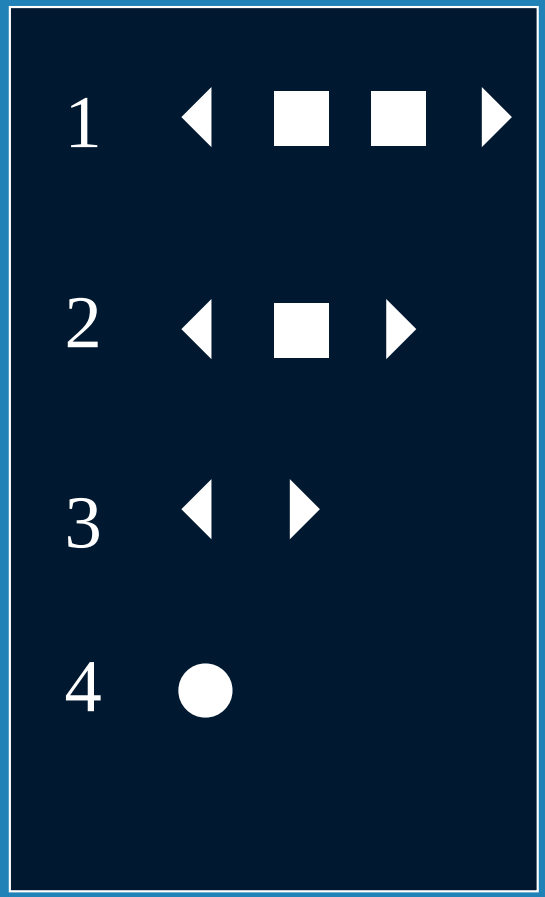
4  Submarines

Battleship Rules

- Ships can be oriented horizontally or vertically.
- Ships may not intersect or share adjacent grid cells, even diagonally.
- Digits in rows and columns represent total number of grid cells occupied by vessels.

Example Grid





Objects

Rows: i in {1,...,10}

Columns: j in {1,...,10}

Parts: { ◀ , ▶ , ▲ , ▼ , ■ , ● }

Direction: across (0), down (1)

Ship No.: k in {1, 2, 3, 4}

Variables

- Filled: f(i,j) = i,j contains ship
- Part: pa(i,j) = part a appears at i,j
- Battleship: bi(i), bj(j), bd encodes the position & direction of ship
- Cruisers: ci(k,i), cj(k, j), cd(k) ditto
- Ditto for Destroyers and Submarines. No. of vars?

Initial Board

Ship parts encoded via pa, pb, pc:

	pa	pb	pc
◂	0	0	0
▸	0	0	1
▲	0	1	0
▼	0	1	1
■	1	0	0
●	1	0	1

Initial Board Clauses

Example:

f(5,8) ~pa(5,8) ~pb(5,8) pc(5,8)
f(7,2) ~pa(7,2) pb(7,2) pc(7,2)
f(10,5) ~pa(10,5) ~pb(10,5) pc(10,5)

Total constraints: Less than 10x10x4 (400). More like 30.

Footprints

Destroyer #2 at 6,5 going across means no ship part at 5,4.

di(2,6) dj(2,5) ~dd(2) $\rightarrow$ ~f(5,4)

~di(2,6) + ~dj(2,5) + dd(2) + ~f(5,4)

1x2x7x10x18+ 2x2x8x10x15+

3x2x9x10x12+ 4x2x10x10x9

= 17400 constraints

Row, Col. Constraints

Each row and column (20) must have the number of filled squares sum to the number indicated. We can introduce a set of variables (3x20=60) to represent the sum. Then a set of clauses (3x20=60) match the actual total to the desired total.

Computing a Sum

Boolean variables x1,...,xn.

S(i,t) represents if x1+...+xi = t.

$S(1,1) \equiv x1$, $S(1,0) \equiv \sim x1$, $\sim S(1,t)$ for $1 < t \leq n$. For $i > 1$, t

$S(i-1,t) \quad xi \rightarrow \quad S(i,t+1)$

$S(i-1,t) \sim xi \rightarrow \quad S(i,t)$

$\sim S(i-1,t) \quad xi \rightarrow \sim S(i,t+1)$

$\sim S(i-1,t) \sim xi \rightarrow \sim S(i,t)$

Other Options

- Leads to 20x8x9=1440 new variables and 20x4x8x9=5760 new constraints.
- Could implement an efficient arithmetic circuit (Boolean gates).
- If n small, map directly to count.

All Else Empty

- How can you say that any place that's not a boat is water?
- Don't need to: If there were extra boat pieces, the row/col sums would be off.

Practical Considerations

- Ultimately, the formula has 2586 vars and 150k constraints. Zoinks!
- Michail Lagoudakis at Duke simplified it to 306 vars. 28k constraints. Tough, but *much* easier.

Other Things to Try

- 15 puzzle
- * Mastermind
- Cross sums
- N-queens (HW)

What to Learn

Some tricks for generating SAT encodings of search problems.

Homework 3 (due 10/10)

1. Let f=~(x+ ~y(~x+z)). (a) Write out the truth table for f. (b) Convert the truth table to CNF. (c) Show the series of steps DPLL makes while solving the resulting formula. Assume variables chosen for splitting in the order x, y, z.
2. Using the same f from the first part, follow the 3-CNF conversion algorithm to create an equivalent 3-CNF formula.
3. Describe how to encode n-queens as a SAT problem.