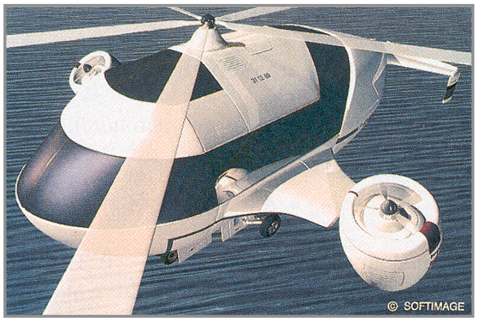
Curved Surfaces

Thomas Funkhouser
Princeton University
C0S 426, Fall 1999

Curved Surfaces

- Motivation
  - Exact boundary representation for some objects
  - More concise representation than polygonal mesh



H&B Figure 10.46

Curved Surfaces

- What makes a good surface representation?
  - Accurate
  - Concise
  - Intuitive specification
  - Local support
  - Affine invariant
  - Arbitrary topology
  - Guaranteed continuity
  - Natural parameterization
  - Efficient display
  - Efficient intersections

Curved Surface Representations

- Polygonal meshes
- Subdivision surfaces
- Parametric surfaces
- Implicit surfaces

Curved Surface Representations

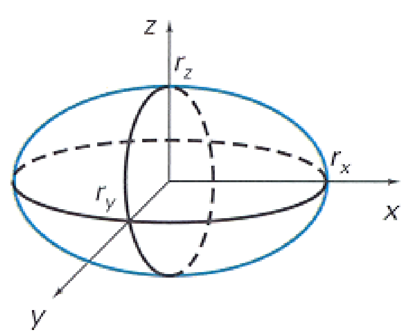
- Polygonal meshes
- Implicit surfaces
- Parametric surfaces
- Subdivision surfaces

Parametric Surfaces

- Boundary defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$

- Example: ellipsoid

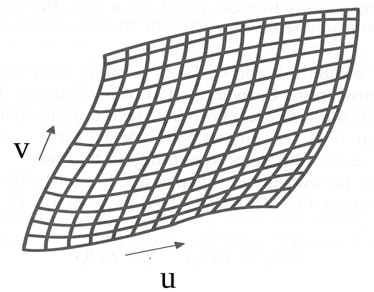
$$x = r_x \cos\phi \cos\theta$$
$$y = r_y \cos\phi \sin\theta$$
$$z = r_z \sin\phi$$



H&B Figure 10.10

Parametric Surfaces

- Advantages:
  - Easy to enumerate points on surface

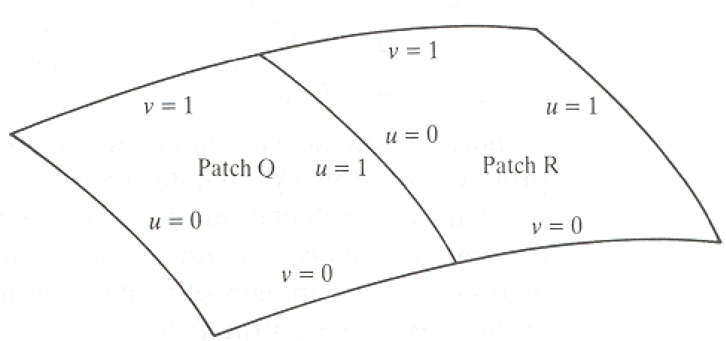


- Problem:
  - Need piecewise-parametrics surfaces to describe complex shapes

FvDFH Figure 11.42

Piecewise Parametric Surfaces

- Surface is partitioned into parametric patches:

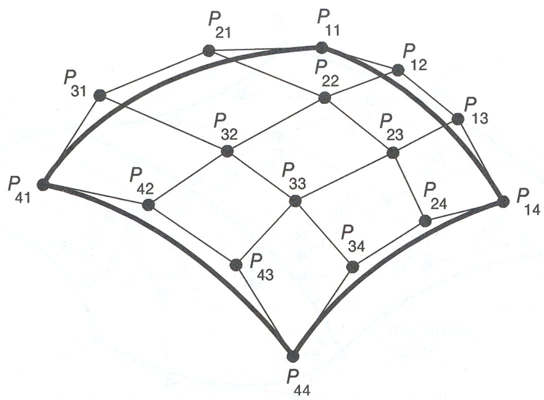


Same ideas as parametric splines!

Watt Figure 6.25

Parametric Patches

- Each patch is defined by blending control points

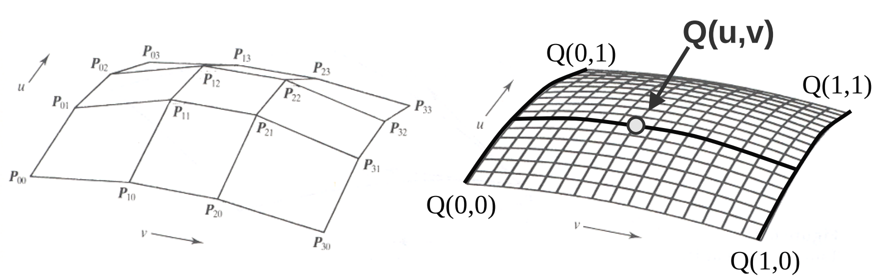


Same ideas as parametric curves!

FvDFH Figure 11.44

Parametric Patches

- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Bicubic Patches

- Point Q(u,v) on any patch is defined by combining control points with polynomial blending functions:

$$Q(u,v) = \mathbf{U}\mathbf{M}\begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix}\mathbf{M}^{\mathbf{T}}\mathbf{V}^{\mathbf{T}}$$

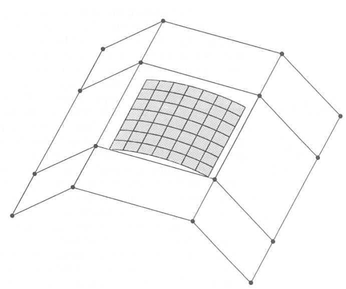
$$\mathbf{U} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} v^3 & v^2 & v & 1 \end{bmatrix}$$

Where M is a matrix describing the blending functions for a parametric cubic curve (e.g., Bezier, B-spline, etc.)

B-Spline Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\mathbf{B-Spline}}\begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix}\mathbf{M}^{\mathbf{T}}_{\mathbf{B-Spline}}\mathbf{V}$$

$$\mathbf{M}_{\mathbf{B-Spline}} = \begin{bmatrix} -1/6 & 1/2 & -1/2 & 1/6 \\ 1/2 & -1 & 1/2 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 1/6 & 2/3 & 1/6 & 0 \end{bmatrix}$$

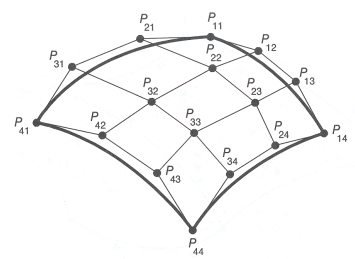


Watt Figure 6.28

Bezier Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\text{Bezier}} \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} \mathbf{M}_{\text{Bezier}}^{\mathbf{T}} \mathbf{V}$$

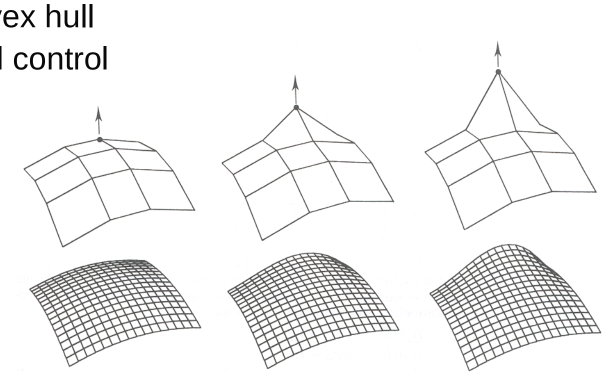
$$\mathbf{M}_{\text{Bezier}} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



FvDFH Figure 11.42

Bezier Patches

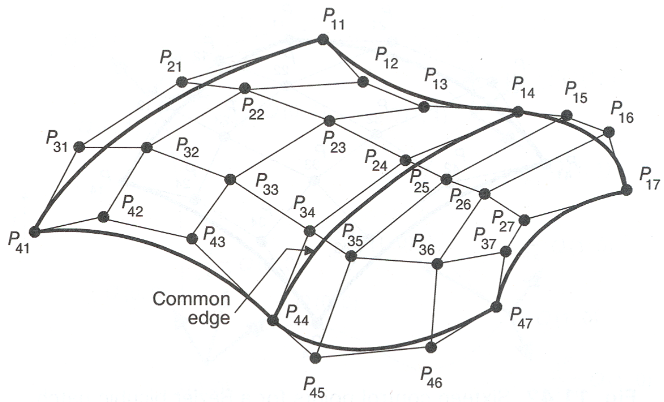
- Properties:
  - Interpolates four corner points
  - Convex hull
  - Local control



Watt Figure 6.22

Bezier Surfaces

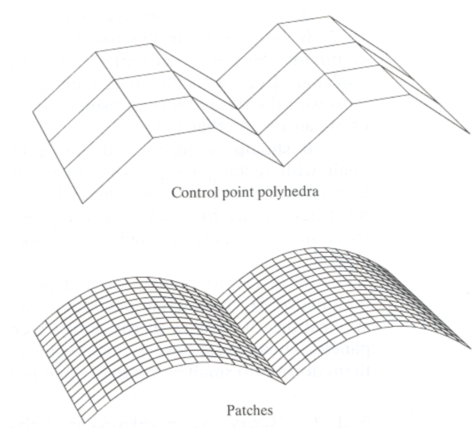
- Continuity constraints are similar to the ones for Bezier splines



FvDFH Figure 11.43

Bezier Surfaces

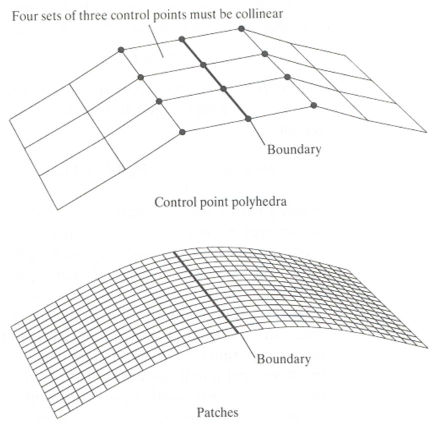
- C^0 continuity requires aligning boundary curves



Watt Figure 6.26a

Bezier Surfaces

- C^1 continuity requires aligning boundary curves and derivatives

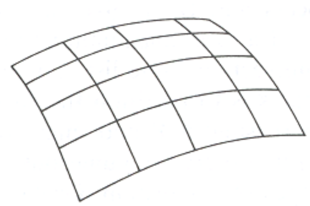


Watt Figure 6.26b

Drawing Bezier Surfaces

- Simple approach is to loop through uniformly spaced increments of u and v

```
DrawSurface(void)
{
    for (int i = 0; i < imax; i++) {
        float u = umin + i * ustep;
        for (int j = 0; j < jmax; j++) {
            float v = vmin + j * vstep;
            DrawQuadrilateral(...);
        }
    }
}
```

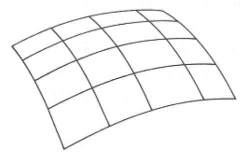


Watt Figure 6.32

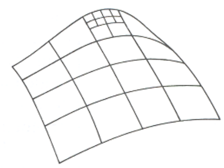
Drawing Bezier Surfaces

- Better approach is to use adaptive subdivision:

```
DrawSurface(surface)
{
    if  Flat (surface, epsilon) {
        DrawQuadrilateral(surface);
    }
    else {
        SubdivideSurface(surface, ...);
        DrawSurface(surfaceLL);
        DrawSurface(surfaceLR);
        DrawSurface(surfaceRL);
        DrawSurface(surfaceRR);
    }
}
```



Uniform subdivision

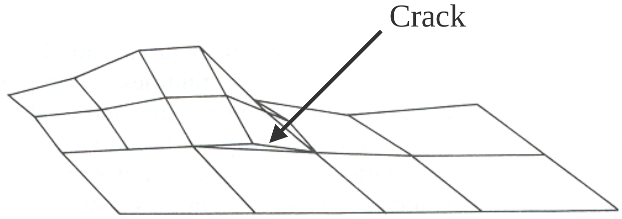


Adaptive subdivision

Watt Figure 6.32

Drawing Bezier Surfaces

- One problem with adaptive subdivision is avoiding cracks at boundaries between patches at different subdivision levels



Avoid these cracks by adding extra vertices and triangulating quadrilaterals whose neighbors are subdivided to a finer level

Watt Figure 6.33

Parametric Surfaces

- Advantages:
  - Easy to enumerate points on surface
  - Possible to describe complex shapes
- Disadvantages:
  - Control mesh must be quadrilaterals
  - Continuity constraints difficult to maintain
  - Hard to find intersections

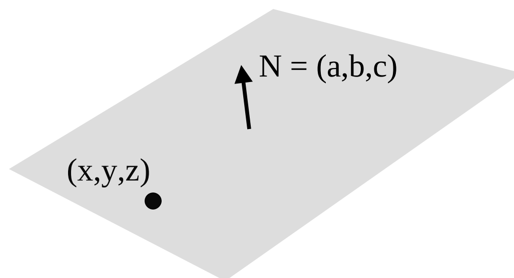
Curved Surface Representations

- Polygonal meshes
- Subdivision surfaces
- Parametric surfaces
- Implicit surfaces

Implicit Surfaces

- Boundary defined by implicit function:
  - $f(x, y, z) = 0$

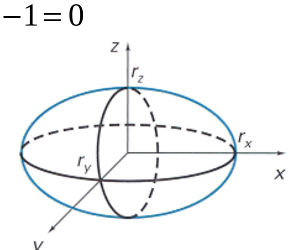
- Example: linear (plane)
  - $ax + by + cz + d = 0$



Implicit Surfaces

- Example: quadric
  - $f(x,y,z)=ax^2+by^2+cz^2+2dxy+2eyz+2fxz+2gx+2hy+2jz+k$

- Common quadric surfaces:
  - Sphere
  - Ellipsoid $\longrightarrow$ $\left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{z}{r_z}\right)^2 - 1 = 0$
  - Torus
  - Paraboloid
  - Hyperboloid



H&B Figure 10.10

Implicit Surfaces

- Advantages:
  - Easy to test if point is on surface
  - Easy to intersect two surfaces
  - Easy to compute z given x and y
- Disadvantages:
  - Hard to describe complex shapes
  - Hard to enumerate points on surface

Summary

Feature	Polygonal Mesh	Implicit Surface	Parametric Surface	Subdivision Surface
Accurate	No	Yes	Yes	Yes
Concise	No	Yes	Yes	Yes
Intuitive specification	No	No	Yes	No
Local support	Yes	No	Yes	Yes
Affine invariant	Yes	Yes	Yes	Yes
Arbitrary topology	Yes	No	No	Yes
Guaranteed continuity	No	Yes	Yes	Yes
Natural parameterization	No	No	Yes	No
Efficient display	Yes	No	Yes	Yes
Efficient intersections	No	Yes	No	No