COS 341, December 13, 2000
Handout Number 9

The following are last year’s final exam problems and their solutions. Keep in mind that, in contrast to the current year, the exam last year was a take-home exam.

Last Year’s Final Exam Problems

In the following, \(i, j, k, n\) take on integer values only.

Problem 1 [20 points] Let \(b_0, b_1, b_2, \ldots\) be the sequence defined by the following recurrence relation:

\[
\begin{align*}
  b_0 &= 1, \\
  b_1 &= 2, \\
  b_n &= b_{n-1} + \sum_{1 \leq k \leq n-1} b_k b_{n-1-k} \text{ for } n \geq 2.
\end{align*}
\]

Let \(B(x) = \sum_{k \geq 0} b_k x^k\). Derive a closed-form formula for \(B(x)\).

Problem 2 [20 points] For any integer \(n > 0\), let \(G_n = (V, E)\) be a graph on \(2n\) vertices, where \(V = \{1, 2, 3, \ldots, 2n\}\), and \(E = \{\{i, j\} \mid 1 \leq i < j \leq 2n, j \neq n + i\}\). Answer the following questions, each with a concise but rigorous justification.

(a) For what values of \(n\) are \(G_n\) Eulerian?
(b) For what values of \(n\) do \(G_n\) contain a Hamiltonian circuit?
(c) What is \(\omega(G_n)\), the size of the largest clique in \(G_n\)?
(d) What is \(\chi(G_n)\), the chromatic number of \(G_n\)?

Remarks In other words, \(G_n\) is obtained from the complete graph on \(2n\) vertices by deleting \(n\) edges (no two of which have any endpoints in common). Thus, \(G_n\) has exactly \(\binom{2n}{2} - n\) edges.

Problem 3 [20 points] For any integer \(n > 0\), let \(H_n = (V, E)\) be a graph on \(4n + 1\) vertices, where \(V = \{1, 2, 3, \ldots, 4n, 4n + 1\}\), and

\[
E = \{\{i, i+1\} \mid 1 \leq i \leq 4n-1\} \cup \{\{1, 4n\}, \{4n+1, n\}, \{4n+1, 2n\}, \{4n+1, 3n\}, \{4n+1, 4n\}\}.
\]

Thus, \(H_n\) has exactly \(4n + 4\) edges. Let \(s_n\) be the number of spanning trees for \(H_n\). Determine \(s_n\) as a closed-form expression of \(n\).
Solutions

Problem 1

\[
\sum_{n \geq 2} b_n x^n = \sum_{n \geq 2} b_{n-1} x^n + \sum_{n \geq 2} x^n \sum_{1 \leq k \leq n-1} b_k b_{n-1-k} = x \sum_{n \geq 2} b_{n-1} x^{n-1} + x \sum_{m \geq 1} x^m \sum_{1 \leq k \leq m} b_k b_{m-k}.
\]

This implies

\[
B(x) - b_0 - b_1 x = x(B(x) - b_0) + x(b_1 x + b_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots),
\]

ie,

\[
B(x) - 1 - 2x = x(B(x) - 1) + x(B(x) - 1)B(x).
\]

This leads to \(x B(x)^2 - B(x) + (1 + x) = 0\), and hence using \(B(0) = b_0 = 1\) we have

\[
B(x) = \frac{1 - \sqrt{1 - 4x(1 + x)}}{2x}.
\]

Problem 2

(a) For \(n = 1\), \(G_n\) consists of two isolated vertices and is thus by definition Eulerian. For \(n > 1\), \(G_n\) is Eulerian since (A) it is connected (vertex 1 is connected to vertex \(n + 1\) through \(1 - 2 - (n + 1)\), and vertex 1 has an edge to each of the remaining vertices) and (B) every vertex has even degree (in fact \(2n - 2\)).

(b) For \(n = 1\), \(G_n\) consists of two isolated vertices and has no Hamiltonian circuit. For \(n > 1\), \(G_n\) has the following Hamiltonian circuit 1, 2, 3, \(\ldots\), \(n - 1\), \(n\), \(n + 1\), \(n + 2\), \(\cdots\), \(2n - 1\), 1.

(c) The answer is \(\omega(G_n) = n\). Note that \(\omega(G_n) \geq n\) since \(\{1, 2, \ldots, n\}\) is a clique; \(\omega(G_n) < n + 1\) since any clique can contain at most one of the vertices \(i, n + i\) for each \(1 \leq i \leq n\).

(d) The answer is \(\chi(G_n) = n\). Note that \(\chi(G_n) \geq n\) since \(\{1, 2, \ldots, n\}\) is a clique and thus each vertex in it has to be painted with a different color; \(\chi(G_n) \leq n\) since we can just paint both vertices \(i, n + i\) with color \(i\), for each \(1 \leq i \leq n\).

Problem 3 Let \(E_0 = \{\{4n + 1, n\}, \{4n + 1, 2n\}, \{4n + 1, 3n\}, \{4n + 1, 4n\}\}\), and

\[
E_1 = \{\{4n, 1\}, \{1, 2\}, \{2, 3\}, \cdots, \{n - 1, n\}\},
\]

\[
E_2 = \{\{n, n + 1\}, \{n + 1, n + 2\}, \{n + 2, n + 3\}, \cdots, \{2n - 1, 2n\}\},
\]

\[
E_3 = \{\{2n, 2n + 1\}, \{2n + 1, 2n + 2\}, \{2n + 2, 2n + 3\}, \cdots, \{3n - 1, 3n\}\},
\]

\[
E_4 = \{\{3n, 3n + 1\}, \{3n + 1, 3n + 2\}, \{3n + 2, 3n + 3\}, \cdots, \{4n - 1, 4n\}\}.
\]
Then \( E = \bigcup_{0 \leq i \leq 4} E_i \).

A spanning tree of \( H_n \) has \( 4n \) edges, and can be specified by the 4 edges missing from \( E \). For \( \alpha \in \{0, 1, 2, 3, 4\} \), let \( s_{n, \alpha} \) be the number of spanning trees of \( H_n \) for which \( \alpha \) of the missing edges are from \( E_0 \). Then

\[
s_n = \sum_{0 \leq \alpha \leq 4} s_{n, \alpha}.
\]

Clearly, \( s_{n, 4} = 0 \) since at least one edge from \( E_0 \) is needed to keep vertex \( 4n + 1 \) from being isolated.

To calculate \( s_{n, 3} \), we count first how many spanning trees there are that contain \( \{4n + 1, n\} \) but no other edge from \( E_0 \). A spanning tree is now specified by the one missing edge from \( \bigcup_{1 \leq i \leq 4} E_i \), so that number is \( | \bigcup_{1 \leq i \leq 4} E_i | = 4n \). We can prove the same result if we count the number of spanning trees that contain any one specific edge but no other edges in \( E_0 \). Thus,

\[
s_{n, 3} = 4 \cdot 4n = 16n.
\]

To calculate \( s_{n, 2} \), let \( a_n \) be the number of spanning trees containing \( \{4n + 1, n\}, \{4n + 1, 2n\} \) but no other edges in \( E_0 \); let \( b_n \) be the number of spanning trees containing \( \{4n + 1, n\}, \{4n + 1, 3n\} \) but no other edges in \( E_0 \). Clearly,

\[
s_{n, 2} = 4a_n + 2b_n.
\]

We compute \( a_n \). A spanning tree of this type is specified by a missing edge chosen from \( E_2 \), and a missing edge from \( E_1 \cup E_3 \cup E_4 \). Thus,

\[
a_n = |E_2| \cdot |E_1 \cup E_3 \cup E_4| = 3n^2.
\]

Similarly,

\[
b_n = |E_2 \cup E_3| \cdot |E_1 \cup E_4| = 4n^2.
\]

This leads to

\[
s_{n, 2} = 4 \cdot 3n^2 + 2 \cdot 4n^2 = 20n^2.
\]

To calculate \( s_{n, 1} \), let \( c_n \) be the number of spanning trees containing \( \{4n + 1, n\}, \{4n + 1, 2n\}, \{4n + 1, 3n\} \) but no other edges in \( E_0 \). Then \( s_{n, 1} = 4c_n \). To compute \( c_n \), note that such a spanning tree is specified by a missing edge from each of the sets \( E_2, E_3, E_1 \cup E_1 \). Thus, \( c_n = |E_2| \cdot |E_3| \cdot |E_1 \cup E_1| = 2n^3 \). Hence,

\[
s_{n, 1} = 4 \cdot 2n^3 = 8n^3.
\]
To calculate $s_{n,0}$, note that such a spanning tree is specified by a missing edge from each of the sets $E_1, E_2, E_3, E_4$. Thus,

$$s_{n,0} = |E_1| \cdot |E_2| \cdot |E_3| \cdot |E_4| = n^4.$$ 

Putting everything together, we have

$$s_n = \sum_{0 \leq \alpha \leq 4} s_{n,\alpha} = n^4 + 8n^3 + 20n^2 + 16n.$$