All proofs in the paper and this appendix are formalized in Coq. The address is: https://github.com/QinxiangCao/UnifySL.

A Proof of Semantic Equivalence

Here, we recall the definition of downwards closure and upwards closure.

Definition 1 (Upwards closure and downwards closure). *Given a separation algebra $(M, \leq, \oplus)$, its upwards closure is the triple $(M, \leq, \oplus^{\Uparrow})$ where $\oplus^{\Uparrow}(m_1, m_2, m)$ iff there is m' such that $m' \leq m$ and $\oplus(m_1, m_2, m')$.*

Given a separation algebra $(M, \leq, \oplus)$, its downwards closure is the triple $(M, \leq, \oplus^{\Downarrow})$ where $\oplus^{\Downarrow}(m_1, m_2, m)$ iff there are m_1' and m_2' such that $m_1 \leq m_1'$, $m_2 \leq m_2'$ and $\oplus(m_1', m_2', m)$.

We first prove that both upwards closures of downwards-closed algebra and downwards closures of upwards-closed algebra are upwards-closed and downwards-closed at the same time. We then prove that the flat semantics on the closures are equivalent with downwards (resp. upwards) semantics in the original algebra.

Lemma 1. *Given an ordered separation algebra $(M, \leq, \oplus)$: If $(M, \leq, \oplus)$ is downwards closed, $(M, \leq, \oplus^{\Uparrow})$ is an upwards closed and downwards closed ordered separation algebra. If $(M, \leq, \oplus)$ is upwards closed, $(M, \leq, \oplus^{\Downarrow})$ is an upwards closed and downwards closed ordered separation algebra.*

Proof.

1. If $(M, \leq, \oplus)$ is upwards-closed, $(M, \leq, \oplus^{\Downarrow})$ is downwards-closed: Obvious because $\leq$ is transitive.
2. If $(M, \leq, \oplus)$ is upwards-closed, $(M, \leq, \oplus^{\Downarrow})$ is upwards-closed: Suppose m_1, m_2, m and n satisfy $\oplus^{\Downarrow}(m_1, m_2, m)$ and $m \leq n$. According to the definition of $\oplus^{\Downarrow}$, there exist m_1' and m_2' s.t.

$$m_1 \leq m_1'$$
$$m_2 \leq m_2'$$
$$\oplus\,(m_1', m_2', m)$$

   Since $(M, \leq, \oplus)$ is upwards-closed, there must exist n_1 and n_2 s.t.

$$m_1' \leq n_1$$
$$m_2' \leq n_2$$
$$\oplus\,(n_1, n_2, n)$$

   So we know that $m_1 \leq n_1$, $m_2 \leq n_2$ and $\oplus^{\Downarrow}(n_1, n_2, n)$. In other words, $(M, \leq, \oplus^{\Downarrow})$ is upwards-closed.
3. If $(M, \leq, \oplus)$ is downwards-closed, $(M, \leq, \oplus^{\Uparrow})$ is upwards-closed: Obvious because $\leq$ is transitive.

4. If $(M, \leq, \oplus)$ is downwards-closed, $(M, \leq, \oplus^{\Uparrow})$ is downwards-closed: Suppose m_1, m_2, m, n_1 and n_2 satisfy $\oplus^{\Uparrow}(m_1, m_2, m)$, $n_1 \leq m_1$ and $n_2 \leq m_2$. According to the definition of $\oplus^{\Uparrow}$, there exist m' s.t.

$$m' \leq m$$
$$\oplus (m_1, m_2, m')$$

Since $(M, \leq, \oplus)$ is downwards-closed, there exist n s.t.

$$n \leq m'$$
$$\oplus (n_1, n_2, n)$$

So we know that $n \leq m$ and $\oplus^{\Uparrow}(n_1, n_2, n)$. In other words, $(M, \leq, \oplus^{\Uparrow})$ is downwards-closed.

□

Theorem 1. *thm:updownflat Given an extended Kripke model $\mathcal{M} = (M, \leq, \oplus, J)$*

1. *if it is downwards closed, then the flat semantics on $\mathcal{M}^{\Uparrow}$ is equivalent to the downwards semantics on $\mathcal{M}$, i.e. for any φ and m, $m \models^{=}_{\mathcal{M}^{\Uparrow}} \varphi$ iff $m \models^{\Downarrow}_{\mathcal{M}} \varphi$*
2. *if it is upwards closed, then the flat semantics on $\mathcal{M}^{\Downarrow}$ is equivalent to the upwards semantics on $\mathcal{M}$, i.e. for any φ and m, $m \models^{=}_{\mathcal{M}^{\Downarrow}} \varphi$ iff $m \models^{\Uparrow}_{\mathcal{M}} \varphi$*

Proof. We prove it by induction on the syntax of assertions. The only interesting cases are the following:

1. If $(M, \leq, \oplus)$ is downwards-closed and for any m:

$$m \models^{=}_{\mathcal{M}^{\Uparrow}} \varphi \text{ iff } m \models^{\Downarrow}_{\mathcal{M}} \varphi$$
$$m \models^{=}_{\mathcal{M}^{\Uparrow}} \psi \text{ iff } m \models^{\Downarrow}_{\mathcal{M}} \psi \qquad \text{(IH)}$$

then for any m:

$$m \models^{=}_{\mathcal{M}^{\Uparrow}} \varphi * \psi \text{ iff } m \models^{\Downarrow}_{\mathcal{M}} \varphi * \psi$$

Obvious by IH and the definition of $\oplus^{\Uparrow}$, flat semantics, downwards semantics.

2. If $(M, \leq, \oplus)$ is downwards-closed and for any m:

$$m \models^{=}_{\mathcal{M}^{\Uparrow}} \varphi \text{ iff } m \models^{\Downarrow}_{\mathcal{M}} \varphi$$
$$m \models^{=}_{\mathcal{M}^{\Uparrow}} \psi \text{ iff } m \models^{\Downarrow}_{\mathcal{M}} \psi \qquad \text{(IH)}$$

then for any m:

$$m \models^{=}_{\mathcal{M}^{\Uparrow}} \varphi -\!\!* \psi \text{ iff } m \models^{\Downarrow}_{\mathcal{M}} \varphi -\!\!* \psi$$

Left to right is obvious because $\oplus^{\Uparrow} \supseteq \oplus$. We only prove the other direction here. Suppose $m \models^{\Downarrow}_{\mathcal{M}} \varphi -\!\!* \psi$ (left side), $\oplus^{\Uparrow}(m, m_1, m_2)$ and $m_1 \models^{=}_{\mathcal{M}^{\Uparrow}} \varphi$ (the assumption of right side). Then by the definition of $\oplus^{\Uparrow}$, we know there exists n_2, s.t.

$$n_2 \leq m_2$$
$$\oplus (m, m_1, n_2)$$

And by IH, we know that

$$m_1 \vDash^{\Downarrow}_{\mathcal{M}} \varphi$$

From the fact that $m \vDash^{\Downarrow}_{\mathcal{M}} \varphi \mathrel{-\!\!*} \psi$, we know

$$n_2 \vDash^{\Downarrow}_{\mathcal{M}} \psi$$

Since the denotation of all assertions are monotonic, $m_2 \vDash^{\Downarrow}_{\mathcal{M}} \psi$, i.e. by IH: $m_2 \vDash^{=}_{\mathcal{M}^{\Uparrow}} \psi$

3. If $(M, \leq, \oplus)$ is upwards-closed and for any m:

$$m \vDash^{=}_{\mathcal{M}^{\Downarrow}} \varphi \text{ iff } m \vDash^{\Uparrow}_{\mathcal{M}} \varphi$$
$$m \vDash^{=}_{\mathcal{M}^{\Downarrow}} \psi \text{ iff } m \vDash^{\Uparrow}_{\mathcal{M}} \psi \tag{IH}$$

then for any m:

$$m \vDash^{=}_{\mathcal{M}^{\Downarrow}} \varphi * \psi \text{ iff } m \vDash^{\Uparrow}_{\mathcal{M}} \varphi * \psi$$

Right to left is obvious, because $\oplus \subseteq \oplus^{\Downarrow}$. We only prove the other direction here. Suppose $m \vDash^{=}_{\mathcal{M}^{\Downarrow}} \varphi$, then there exist m_1 and m_2 s.t.

$$\oplus^{\Downarrow} (m_1, m_2, m)$$
$$m_1 \vDash^{=}_{\mathcal{M}^{\Downarrow}} \varphi$$
$$m_2 \vDash^{=}_{\mathcal{M}^{\Downarrow}} \psi$$

By the definition of $\oplus^{\Downarrow}$, there exist n_1 and n_2 s.t.

$$m_1 \leq n_1$$
$$m_2 \leq n_2$$
$$\oplus (n_1, n_2, m)$$

By the monotonicity of φ and ψ's denotation and IH, we know that

$$n_1 \vDash^{\Uparrow}_{\mathcal{M}} \varphi$$
$$n_2 \vDash^{\Uparrow}_{\mathcal{M}} \psi$$

So, $m \vDash^{\Uparrow}_{\mathcal{M}} \varphi * \psi$.

4. If $(M, \leq, \oplus)$ is upwards-closed and for any m:

$$m \vDash^{=}_{\mathcal{M}^{\Downarrow}} \varphi \text{ iff } m \vDash^{\Uparrow}_{\mathcal{M}} \varphi$$
$$m \vDash^{=}_{\mathcal{M}^{\Downarrow}} \psi \text{ iff } m \vDash^{\Uparrow}_{\mathcal{M}} \psi \tag{IH}$$

then for any m:

$$m \vDash^{=}_{\mathcal{M}^{\Downarrow}} \varphi \mathrel{-\!\!*} \psi \text{ iff } m \vDash^{\Uparrow}_{\mathcal{M}} \varphi \mathrel{-\!\!*} \psi$$

Obvious by IH and the definition of $\oplus^{\Downarrow}$, flat semantics, upwards semantics. $\square$

B Proof of Completeness

Here, we always assume Σ is countable and thus so is $\mathcal{L}(\Sigma)$.

We use the standard notation $\Phi \vdash \psi$ to represent the existence of finite elements $\varphi_1, \varphi_2, ..., \varphi_n \in \Phi$ such that

$$\vdash \left(\bigwedge_{i=1}^{n} \varphi_i \right) \to \psi$$

Also, $\Phi \mid\models_U \psi$ means that for any model $m \in U$ if every assertion in Φ is satisfied on m then ψ is satisfied on m.

Definition 2 (Soundness and completeness). *A proof theory Γ is sound w.r.t. a semantics $\models$ in a class of models U iff for any φ, $\vdash^{\Gamma} \varphi$ implies $\mid\models_U \varphi$.*

A proof theory Γ is weakly complete w.r.t. a semantics $\models$ in a class of models U iff for any φ, $\mid\models_U \varphi$ implies $\vdash^{\Gamma} \varphi$.

A proof theory Γ is strongly complete w.r.t. a semantics $\models$ in U iff for any Φ and φ, $\Phi \mid\models_U \varphi$ implies $\Phi \vdash^{\Gamma} \varphi$.

Strong completeness clearly implies weak completeness. In the rest of this appendix we present the soundness and strong completeness of separation logics stated as follows:

Theorem 2 (Parametric soundness and completeness). *A separation logic Γ is sound and strongly complete w.r.t. the flat semantics in Γ's corresponding class of models.*

We first define DDCS and canonical model.

Definition 3 (DDCS). *Given Σ, we call a set of formulas $\Phi \subseteq \mathcal{L}(\Sigma)$ a **d**erivable closed, **d**isjunction-witnessed, **c**onsistent **s**et (DDCS) of proof theory Γ if*

1. *it is* derivable closed*, i.e. for any ϕ, if $\Phi \vdash^{\Gamma} \phi$ then $\phi \in \Phi$*
2. *it is* disjunction witnessed*, i.e. for any ϕ and ψ, if $\phi \vee \psi \in \Phi$ then $\phi \in \Phi$ or $\psi \in \Phi$*
3. *it is* consistent*, i.e. $\Phi \nvdash^{\Gamma} \bot$*

Definition 4. *We lift separating conjunction to sets of assertions (not only DDCSs but any sets), as follows:*

$$\Phi * \Psi \triangleq \{\phi * \psi \mid \Phi \vdash^{\Gamma} \phi \text{ and } \Psi \vdash^{\Gamma} \psi\}$$

Definition 5 (Canonical model). *Given a separation logic Γ of $\mathcal{L}(\Sigma)$, we call $\mathcal{M}^c = (M^c, \leq^c, \oplus^c, J^c)$ the canonical model of Γ where*

1. *M^c is the set of DDCSs of Γ*
2. *for any $\Phi, \Psi \in M^c$, $\Phi \leq^c \Psi$ iff $\Phi \subseteq \Psi$*
3. *for any $\Phi_1, \Phi_2, \Phi \in M^c$, $\oplus^c(\Phi_1, \Phi_2, \Phi)$ iff $\Phi_1 * \Phi_2 \subseteq \Phi$*
4. *for any $p \in \Sigma$ and $\Phi \in M^c$, $\Phi \in J^c(p)$ iff $p \in \Phi$*

It is an important fact that for any DDCS Φ of Γ and assertion φ, $\Phi \vdash^{\Gamma} \varphi$ is equivalent with $\varphi \in \Phi$. We will use these two concept interchangably in the following proofs.

Here, we first prove a lemma about the lifted separating conjunction between sets of assertions.

Lemma 2. *For any Φ, Ψ and θ, if $\Phi * \Psi \vdash \chi$ then there exist $\varphi_1, \varphi_2, .., \varphi_n \in \Phi$ and $\psi_1, \psi_2, ..., \psi_m \in \Psi$, s.t.*

$$\vdash \left(\bigwedge_{i=1}^{n} \varphi_i * \bigwedge_{i=1}^{m} \psi_i \right) \to \chi$$

Proof. By the definition of "derivable", we know there exist

$$\chi_1, \chi_2, .., \chi_k, \quad \chi'_1, \chi'_2, .., \chi'_k, \quad \chi''_1, \chi''_2, .., \chi''_k$$
$$\varphi_{1,1}, ..., \varphi_{1,n_1}, \varphi_{2,1}, ..., \varphi_{2,n_2}, ..., \varphi_{k,1}, ..., \varphi_{k,n_k} \in \Phi$$
$$\psi_{1,1}, ..., \psi_{1,m_1}, \psi_{2,1}, ..., \psi_{2,m_2}, ..., \psi_{k,1}, ..., \psi_{k,m_k} \in \Psi$$

such that

$$\vdash \left(\bigwedge_{j=1}^{n_1} \varphi_{1,j} \right) \to \chi'_1, \quad \vdash \left(\bigwedge_{j=1}^{m_1} \psi_{1,j} \right) \to \chi''_1$$
$$\vdash \left(\bigwedge_{j=1}^{n_2} \varphi_{2,j} \right) \to \chi'_2, \quad \vdash \left(\bigwedge_{j=1}^{m_2} \psi_{2,j} \right) \to \chi''_2$$
$$...$$
$$\vdash \left(\bigwedge_{j=1}^{n_k} \varphi_{k,j} \right) \to \chi'_k, \quad \vdash \left(\bigwedge_{j=1}^{m_k} \psi_{k,j} \right) \to \chi''_k$$
$$\vdash \left(\bigwedge_{i=1}^{k} \chi'_i * \chi''_i \right) \to \chi$$

By the fact the following assertion is a tautology in any separation logic for any φ, φ', ψ and ψ':

$$(\varphi \wedge \varphi') * (\psi \wedge \psi') \to (\varphi * \psi) \wedge (\varphi' * \psi')$$

we know that

$$\vdash \left(\bigwedge_{i=1}^{k} \chi'_i \right) * \left(\bigwedge_{i=1}^{k} \chi''_i \right) \to \chi$$

So, by $*$MONO, we know

$$\vdash \left(\bigwedge_{i=1}^{k} \bigwedge_{j=1}^{n_j} \varphi_{i,j} \right) * \left(\bigwedge_{i=1}^{k} \bigwedge_{j=1}^{m_j} \psi_{i,j} \right) \to \chi \tag{1}$$

□

One important property of DDCS's is that we can extend consistent set of assertions into DDCS's. Specifically, we prove the following two existence lemma.

Lemma 3 (Existence lemma I). *Given a separation logic Γ, a set of assertions Φ and an assertion ϕ, if $\Phi \nvdash^{\Gamma} \phi$, then there exists Φ', which is a DDCS of Γ, s.t. $\Phi \subseteq \Phi'$ and $\Phi' \nvdash^{\Gamma} \phi$.*

Proof. See [1]. □

Lemma 4 (Existence lemma II). *Given a separation logic Γ, of $\mathcal{L}(\Sigma)$, and sets of assertions Φ_1, Φ_2 and Φ among which Φ is a DDCS of Γ, if $\Phi_1 * \Phi_2 \subseteq \Phi$, then*

1. *there is a DDCS Φ_1', s.t. $\Phi_1 \subseteq \Phi_1'$ and $\Phi_1' * \Phi_2 \subseteq \Phi$.*
2. *there is a DDCS Φ_2', s.t. $\Phi_2 \subseteq \Phi_2'$ and $\Phi_1 * \Phi_2' \subseteq \Phi$.*

Proof. We only prove the first half here. The second half follows in the same way.

Because $\mathcal{L}(\Sigma)$ is countable, we can enumerate all asserts as $\psi_1, \psi_2, ...$ Then we constructor the following sets of assertions and let $\Phi_1' \triangleq \bigcup_{k\in\mathbb{N}} \Psi_k$:

$$\Psi_0 = \Phi_1$$
$$\Psi_{k+1} = \begin{cases} \Psi_k \cup \{\psi_{k+1}\} & \text{if } (\Psi_k \cup \{\psi_{k+1}\}) * \Phi_2 \subseteq \Phi \\ \Psi_k & \text{otherwise} \end{cases}$$

Obviously, $\Phi_1 \subseteq \Phi_1'$ and $\Phi_1' * \Phi_2 \subseteq \Phi$, so we only need to show that Φ_1' is actually a DDCS.

First, we prove that Φ_1' is derivable closed by contradiction. Suppose $\Phi_1' \vdash^{\Gamma} \psi_k$ and $\psi_k \notin \Phi_1'$. Then $\psi_k \notin \Psi_k$, which means there exists φ_1 and φ_2 s.t.

$$\Psi_{k-1} \cup \{\psi_k\} \vdash^{\Gamma} \varphi_1$$
$$\Phi_2 \vdash^{\Gamma} \varphi_2$$
$$\varphi_1 * \varphi_2 \notin \Phi \tag{2}$$

At the same time, we know that $\Phi_1' \vdash^{\Gamma} \varphi_1$ because $\Psi_k \subseteq \Phi_1'$ and $\Phi_1' \vdash^{\Gamma} \psi_k$. Since we know $\Phi_1' * \Phi_2 \subseteq \Phi$, we can conclude that $\varphi_1 * \varphi_2 \in \Phi$, which contradicts (2)!

Second, Φ_1' is disjunction-witnessed. We prove it by contradiction, mostly in the same way as above. Suppose $\psi_k \vee \psi_{k'} \in \Phi_1'$, $\psi_k \notin \Phi_1'$ and $\psi_{k'} \notin \Phi_1'$. Then $\psi_k \notin \Psi_k$ and $\psi_{k'} \notin \Psi_{k'}$, which means there exists φ_1, φ_2, φ_1' and φ_2' s.t.

$$\Psi_{k-1} \cup \{\psi_k\} \vdash^{\Gamma} \varphi_1 \tag{3}$$
$$\Psi_{k'-1} \cup \{\psi_{k'}\} \vdash^{\Gamma} \varphi_1' \tag{4}$$
$$\Phi_2 \vdash^{\Gamma} \varphi_2$$
$$\Phi_2 \vdash^{\Gamma} \varphi_2'$$
$$\varphi_1 * \varphi_2 \notin \Phi \tag{5}$$
$$\varphi_1' * \varphi_2' \notin \Phi \tag{6}$$

From deduction theorem and (3) (4), we know:

$$\Psi_{k-1} \vdash^{\Gamma} \psi_k \to \varphi_1$$
$$\Psi_{k'-1} \vdash^{\Gamma} \psi_{k'} \to \varphi_1'$$

Then, by $\Phi_1' \supseteq \Psi_{k-1}$ and $\Phi_1' \supseteq \Psi_{k'-1}$, we know

$$\Phi_1' \vdash^{\Gamma} \psi_k \vee \psi_{k'} \to \varphi_1 \vee \varphi_1'$$

Moreoever, since $\psi_k \vee \psi_{k'} \in \Phi_1'$ is assumed, $\Phi_1' \vdash^{\Gamma} \varphi_1 \vee \varphi_1'$. At the same time, $\Phi_2 \vdash^{\Gamma} \varphi_2 \wedge \varphi_2'$ and $\Phi_1' * \Phi_2 \subseteq \Phi$ is already known, so

$$\Phi \vdash (\varphi_1 \vee \varphi_1') * (\varphi_2 \wedge \varphi_2')$$

Notice that the following assertion is a tautology in any separation logic,

$$(\varphi_1 \vee \varphi_1') * (\varphi_2 \wedge \varphi_2') \to (\varphi_1 * \varphi_2 \vee \varphi_1' * \varphi_2')$$

Thus, we know the following fact because Φ is a DDCS:

$$\varphi_1 \vee \varphi_1' \in \Phi \text{ or } \varphi_2 \wedge \varphi_2' \in \Phi$$

which contradicts with (5) and (6)!

Third, Φ_1' is consistent. This follows the facts that Φ is consistent, $\Phi_1' * \Phi_2 \subseteq \Phi$ and $\vdash^{\Gamma} \bot * \top \to \bot$. □

Now we can prove that a canonical model is actually well defined, i.e.we will show that $\leq^c$ is a preorder, $\oplus^c$ is commutative and associative and J^c is monotonic. Also, it is upwards closed and downwards closed.

Lemma 5. *Given a separation logic Γ of $\mathcal{L}(\Sigma)$, its canonical model $(M^c, \leq^c, \oplus^c, J^c)$ is an extended Kripke model, which is upwards closed and downwards closed at the same time.*

Proof.

1. $\leq^c$ is a preorder because set inclusion is preordered.
2. $\oplus^c$ is commutative: suppose Φ_1, Φ_2 and Φ are DDCS's and $\oplus^c(\Phi_1, \Phi_2, \Phi)$. By definition, for any φ_1 and φ_2, if $\Phi_1 \vdash^{\Gamma} \varphi_1$ and $\Phi_2 \vdash^{\Gamma} \varphi_2$ then $\Phi \vdash^{\Gamma} \varphi_1 * \varphi_2$. Since Γ is a separation logic,

$$\vdash^{\Gamma} \varphi_1 * \varphi_2 \to \varphi_2 * \varphi_1$$

and thus $\oplus^c(\Phi_2, \Phi_1, \Phi)$.
3. $\oplus^c$ is associative: suppose $\Phi_x, \Phi_y, \Phi_z, \Phi_{xy}$ and Φ_{xyz} are DDCS's such that $\oplus^c(\Phi_x, \Phi_y, \Phi_{xy})$ and $\oplus^c(\Phi_{xy}, \Phi_z, \Phi_{xyz})$. First we show that

$$\Phi_x * (\Phi_y * \Phi_z) \subseteq \Phi_{xyz}.$$

Given $\varphi_x \in \Phi_x$, $\varphi_y \in \Phi_y$ and $\varphi_z \in \Phi_z$, we know that $\varphi_x * \varphi_y \in \Phi_{xy}$. Thus,

$$(\varphi_x * \varphi_y) * \varphi_z \in \Phi_{xyz} \text{ or } \Phi_{xyz} \vdash^{\Gamma} (\varphi_x * \varphi_y) * \varphi_z$$

From the fact that Γ is a separation logic, it follows that $\Phi_{xyz} \vdash^{\Gamma} \varphi_x * (\varphi_y * \varphi_z)$.
Second, by existence lemma II(2), we know there exists a DDCS Φ_{yz} such that $\Phi_x * \Phi_{yz} \subseteq \Phi_{xyz}$ and $\Phi_y * \Phi_z \subseteq \Phi_{yz}$, which implies that $\oplus^c$ is associative by definition.
4. J^c is monotonic by definition.
5. Upwards-closed: trivial because $\Phi_1 * \Phi_2 \subseteq \Phi$ and $\Phi \subseteq \Phi'$ implies $\Phi_1 * \Phi_2 \subseteq \Phi'$.
6. Downwards-closed: trivial because $\Phi_1 * \Phi_2 \subseteq \Phi$, $\Phi'_1 \subseteq \Phi_1$ and $\Phi'_2 \subseteq \Phi_2$ implies $\Phi'_1 * \Phi'_2 \subseteq \Phi$.

□

Lemma 6. *Given a separation logic Γ of $\mathcal{L}(\Sigma)$, its canonical model $(M^c, \leq^c, \oplus^c, J^c)$ is a unital extended Kripke model.*

Proof. Assume Φ is an arbitary DDCS. Because EMP $\in \Gamma$, we know that

$$\{\mathsf{emp}\} * \Phi \subseteq \Phi$$

By existence lemma II, we know there is a DDCS Ψ such that $\mathsf{emp} \in \Psi$ and $\Psi * \Phi \subseteq \Phi$. So, Ψ is Φ's increasing residual. □

Because a canonical model has a upwards-closed and downwards-closed separation algebra we can define the flat semantics on it. The most important property of canonical models, formalized by the truth lemma below, is that assertions in a DDCS are the assertions that are exactly the ones satisfied on the same DDCS w.r.t. flat semantics.

Lemma 7 (Truth lemma). *Given a separation logic Γ of $\mathcal{L}(\Sigma)$, for any $\Phi \in M^c$ and $\varphi \in \mathcal{L}(\Sigma)$,*

$$\Phi \vDash^{=}_{\mathcal{M}^c} \varphi \quad \textit{iff} \quad \varphi \in \Phi$$

Proof. We proceed by induction on the syntax of φ. The cases when φ is an atomic assertion, a conjunction, a disjunction or an implication, are covered in the proof of intuitionistic logic completeness [1]. Here, we show the base step for emp and the induction step for separating conjunction and separating disjunction. Specifically, we need to show that

$$\Phi \vDash^{=}_{\mathcal{M}^c} \mathsf{emp} \quad \text{iff} \quad \mathsf{emp} \in \Phi \qquad (\text{Cemp})$$

and given the induction hypothesis: for any DDCS Φ,

$$\begin{aligned} \Phi \vDash^{=}_{\mathcal{M}^c} \varphi_1 \quad &\text{iff} \quad \varphi_1 \in \Phi \\ \Phi \vDash^{=}_{\mathcal{M}^c} \varphi_2 \quad &\text{iff} \quad \varphi_2 \in \Phi \end{aligned} \qquad (\text{IH})$$

we are going to show that for any DDCS Φ:

$$\Phi \vDash^{=}_{\mathcal{M}^c} \varphi_1 * \varphi_2 \quad \text{iff} \quad \varphi_1 * \varphi_2 \in \Phi \qquad (\text{C}*)$$

$$\Phi \vDash^{=}_{\mathcal{M}^c} \varphi_1 -\!\!* \varphi_2 \quad \text{iff} \quad \varphi_1 -\!\!* \varphi_2 \in \Phi \qquad (\text{C}-\!\!*)$$

Cemp $\Rightarrow$ Suppose Φ is increasing. Then we first show that $\Phi * \{\mathsf{emp}\} \vdash^{\Gamma} \mathsf{emp}$. If not, then we know from existence lemma I that there exists a DDCS Φ_2 s.t.

$$\begin{aligned} &\Phi * \{\mathsf{emp}\} \vdash^{\Gamma} \subseteq \Phi_2 \\ &\Phi_2 \not\vdash^{\Gamma} \mathsf{emp} \end{aligned} \tag{7}$$

Then, from existence lemma II(2), we know there exists a DDCS Φ_1 s.t.

$$\begin{aligned} &\Phi * \Phi_1 \vdash^{\Gamma} \subseteq \Phi_2 \\ &\mathsf{emp} \in \Phi_1 \end{aligned}$$

Since Φ is increasing, $\mathsf{emp} \in \Phi_1 \subseteq \Phi_2$. It contradicts with (7)!
Now that $\Phi * \{\mathsf{emp}\} \vdash^{\Gamma} \mathsf{emp}$, we know from lemma 2 that there exists φ s.t.

$$\begin{aligned} &\Phi \vdash^{\Gamma} \varphi \\ &\vdash^{\Gamma} \varphi * \mathsf{emp} \rightarrow \mathsf{emp} \end{aligned}$$

By the adjoint property, we know: $\vdash^{\Gamma} \varphi \rightarrow (\mathsf{emp} \mathrel{-\!\!*} \mathsf{emp})$, so

$$\Phi \vdash^{\Gamma} \mathsf{emp} \mathrel{-\!\!*} \mathsf{emp}$$

Consequencely,

$$\Phi \vdash^{\Gamma} (\mathsf{emp} \mathrel{-\!\!*} \mathsf{emp}) * \mathsf{emp}$$

So, $\Phi \vdash^{\Gamma} \mathsf{emp}$, i.e. $\mathsf{emp} \in \Phi$.

Cemp $\Leftarrow$ Because $\mathsf{emp} \in \Phi$. We know, if $\Phi * \Psi \subseteq \Psi'$, then $\{\mathsf{emp}\} * \Psi \subseteq \Psi'$. This tells $\Psi \subseteq \Psi'$ by EMP. As Ψ and Ψ' is arbitarily chosen, Φ is increasing. So, $\Phi \vDash^{=}_{\mathcal{M}^c} \mathsf{emp}$.

C$*$ $\Rightarrow$ follows by definition.

C$*$ $\Leftarrow$ Given Φ, suppose $\varphi_1 * \varphi_2 \in \Phi$. We start by showing

$$\{\varphi_1\} * \{\varphi_2\} \subseteq \Phi.$$

For any ψ_1 and ψ_2, if $\varphi_1 \vdash^{\Gamma} \psi_1$ and $\varphi_2 \vdash^{\Gamma} \psi_2$, then by $*$MONO

$$\vdash^{\Gamma} \varphi_1 * \varphi_2 \rightarrow \psi_1 * \psi_2.$$

Since $\varphi_1 * \varphi_2 \in \Phi$ and Φ is a DDCS, then $\psi_1 * \psi_1 \in \Phi$.
Second, by applying existence lemma II(1) and II(2), we know that there exists DDCS's Φ_1 and Φ_2 such that $\varphi_1 \in \Phi_1$, $\varphi_2 \in \Phi_2$ and $\Phi_1 * \Phi_2 \subseteq \Phi$. So, by IH and the definition of $\oplus^c$, we know that

$$\Phi_1 \vDash^{=}_{\mathcal{M}^c} \varphi_1,\ \Phi_2 \vDash^{=}_{\mathcal{M}^c} \varphi_2,\ \ \oplus^c(\Phi_1, \Phi_2, \Phi)$$

which shows $\Phi \vDash^{=}_{\mathcal{M}^c} \varphi_1 * \varphi_2$.

C$-\!\!*$ $\Rightarrow$ Given Φ, suppose $\Phi \vDash^{=}_{\mathcal{M}^c} \varphi_1 -\!\!* \varphi_2$. We prove $\varphi_1 -\!\!* \varphi_2 \in \Phi$ by considering whether

$$\Phi * \{\varphi_1\} \vdash^{\Gamma} \varphi_2$$

If it holds, then we know from lemma 2 that there exists φ such that $\Phi \vdash^{\Gamma} \varphi$ and $\vdash^{\Gamma} \varphi * \varphi_1 \to \varphi_2$. By $*$ADJ, we know that $\vdash^{\Gamma} \varphi \to (\varphi_1 -\!\!* \varphi_2)$, thus $\Phi \vdash^{\Gamma} \varphi_1 -\!\!* \varphi_2$.

If it doesn't hold, then we can construct a DDCS Φ_2 by existence lemma I, s.t. $\Phi * \{\varphi_1\} \subseteq \Phi_2$ and $\Phi_2 \nvdash^{\Gamma} \varphi_2$. Moreover, we can construct another DDCS Φ_1 by existence lemma II(1), s.t. $\Phi * \Phi_1 \subseteq \Phi_2$ and $\varphi_1 \in \Phi_1$. So, $\oplus^c(\Phi, \Phi_1, \Phi_2)$. And by IH, $\Phi_1 \vDash^{=}_{\mathcal{M}^c} \varphi_1$ and $\Phi_2 \nvDash^{=}_{\mathcal{M}^c} \varphi_2$. However, this contradicts with the assumption that $\Phi \vDash^{=}_{\mathcal{M}^c} \varphi_1 -\!\!* \varphi_2$.

C$-\!\!*$ $\Leftarrow$ follows from the fact that

$$\vdash^{\Gamma} (\varphi_1 -\!\!* \varphi_2) * \varphi_1 \to \varphi_2$$

□

Lemma 8. *Given a separation logic Γ, its canonical model $\mathcal{M}^c$ satisfies the canonical properties of all optional axioms in Γ.*

Proof. It is well known results that

1. $\mathcal{M}^c$ has an identity relation as its preorder if EM $\in \Gamma$
2. $\mathcal{M}^c$ has an non-branching relation as its preorder if GD $\in \Gamma$
3. $\mathcal{M}^c$ has an always-join relation as its preorder if WEM $\in \Gamma$

Besides,

4. $\mathcal{M}^c$ is increasing separation algebra if $*$E $\in \Gamma$. Suppose Φ_1, Φ_2 and Φ are DDCSs and $\oplus^c(\Phi_1, \Phi_2, \Phi)$. Then for any $\varphi_1 \in \Phi_1$, we know $\varphi_1 * \top \in \Phi$ (because $\top \in \Phi_2$). Since $*$E $\in \Gamma$, $\Phi \vdash^{\Gamma} \varphi_1$, i.e. $\varphi_1 \in \Phi$. So, $\Phi_1 \subseteq \Phi$.
5. $\mathcal{M}^c$ has increasing elements self-joining if eDup $\in \Gamma$. Suppose Φ is an increasing DDCS. By truth lemma we know that $\mathsf{emp} \in \Phi$. Now, we only need to show that $\Phi * \Phi \subseteq \Phi$, i.e. for any $\varphi, \psi \in \Phi$, $\varphi * \psi \in \Phi$. Since Φ is a DDCS, we know that $\mathsf{emp} \wedge (\varphi \wedge \psi) \in \Phi$. By eDup, $(\varphi \wedge \psi) * (\varphi \wedge \psi) \in \Phi$. By $*$MONO, $\varphi * \psi \in \Phi$.
6. $\mathcal{M}^c$'s increasing elements can only be split into smaller pieces if eE $\in \Gamma$. Suppose Ψ_1, Ψ_2 and Φ are DDCSs, Φ is increasing and $\Psi_1 * \Psi_2 \subseteq \Phi$. By truth lemma we know that $\mathsf{emp} \in \Phi$. Now we need to show that $\Psi_1 \subseteq \Phi$. Consider any $\varphi \in \Psi_1$, then we know $\varphi * \top \in \Phi$. Thus $\mathsf{emp} \wedge (\varphi * \top) \in \Phi$. By eE, $\varphi \in \Phi$. Since φ is arbitary, $\Psi_1 \subseteq \Phi$.

□

Now we can prove separation logics complete.

Proof. We will prove the contrapositive of strong completeness. Suppose $\Gamma \nvdash^{\Gamma} \varphi$, we know from existence lemma I that there exists a DDCS Ψ such that $\Phi \subseteq \Psi$ and $\Psi \nvdash^{\Gamma} \varphi$.

By truth lemma, we know that $\Psi \vDash^{=}_{\mathcal{M}} \Phi$ and $\Psi \nvDash^{=}_{\mathcal{M}} \varphi$. By lemma 8, we know that the canonical model of Γ is indeed in the corresponding class of extended Kripke models. □

References

1. Saul A. Kripke. Semantical analysis of intuitionistic logic i. In *Studies in Logic and the Foundations of Mathematics 50*, pages 92–130, 1965.