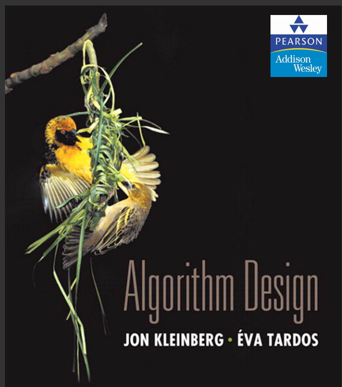


11. Approximation Algorithms

- *load balancing*
- *center selection*
- *pricing method: weighted vertex cover*
- *LP rounding: weighted vertex cover*
- *generalized load balancing*
- *knapsack problem*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on 2/26/20 11:14 AM

Coping with NP-completeness

Q. Suppose I need to solve an **NP**-hard optimization problem.
What should I do?

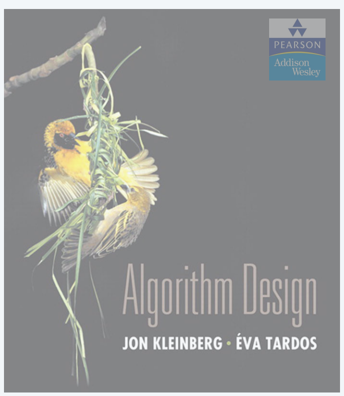
A. Sacrifice one of three desired features.

- i. Runs in polynomial time.
- ii. Solves arbitrary instances of the problem.
- iii. Finds optimal solution to problem.

ρ-approximation algorithm.

- Runs in polynomial time.
- Solves arbitrary instances of the problem
- Finds solution that is within ratio ρ of optimum.

Challenge. Need to prove a solution's value is close to optimum, without even knowing what is optimum value.



Section 11.1

11. Approximation Algorithms

- *load balancing*
- *center selection*
- *pricing method: weighted vertex cover*
- *LP rounding: weighted vertex cover*
- *generalized load balancing*
- *knapsack problem*

Load balancing

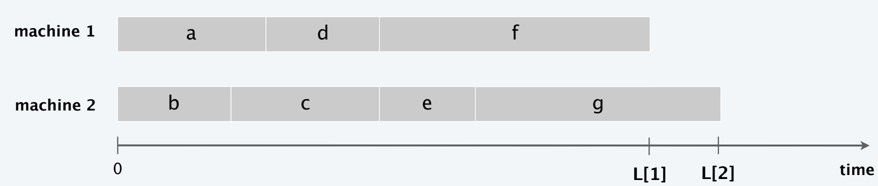
Input. m identical machines; $n \geq m$ jobs, job j has processing time t_j.

- Job j must run contiguously on one machine.
- A machine can process at most one job at a time.

Def. Let $S[i]$ be the subset of jobs assigned to machine i.
The load of machine i is $L[i] = \Sigma_{j \in S[i]}\, t_j$.

Def. The makespan is the maximum load on any machine $L = \max_i L[i]$.

Load balancing. Assign each job to a machine to minimize makespan.

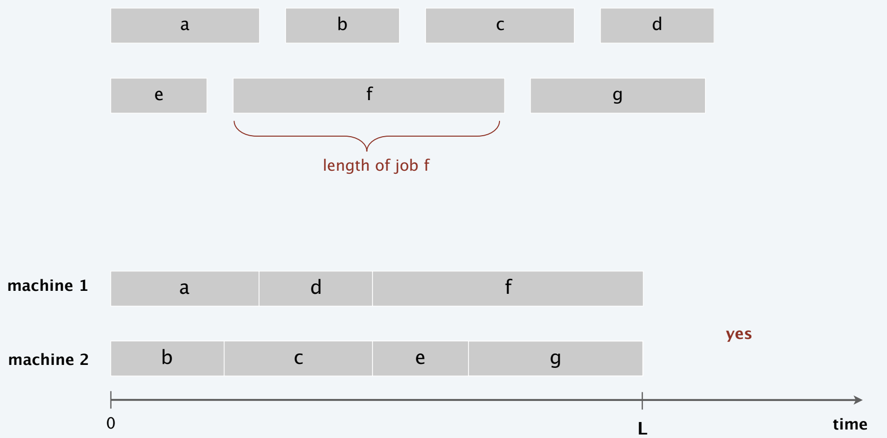


Load balancing on 2 machines is NP-hard

Claim. Load balancing is hard even if $m = 2$ machines.

Pf. PARTITION $\leq_P$ LOAD-BALANCE.

NP-complete by Exercise 8.26



Load balancing: list scheduling

List-scheduling algorithm.

- Consider n jobs in some fixed order.
- Assign job j to machine i whose load is smallest so far.

LIST-SCHEDULING $(m, n, t_1, t_2, \ldots, t_n)$

FOR $i = 1$ TO m

  $L[i] \leftarrow 0.$ ← load on machine i

  $S[i] \leftarrow \varnothing.$ ← jobs assigned to machine i

FOR $j = 1$ TO n

  $i \leftarrow \text{argmin}_k\, L[k].$ ← machine i has smallest load

  $S[i] \leftarrow S[i] \cup \{j\}.$ ← assign job j to machine i

  $L[i] \leftarrow L[i] + t_j.$ ← update load of machine i

RETURN $S[1], S[2], \ldots, S[m].$

Implementation. $O(n \log m)$ using a priority queue for loads $L[k]$.

Load balancing: list scheduling analysis

Theorem. [Graham 1966] Greedy algorithm is a 2-approximation.

- First worst-case analysis of an approximation algorithm.
- Need to compare resulting solution with optimal makespan L^*.

Lemma 1. For all k: the optimal makespan $L^* \geq t_k$.

Pf. Some machine must process the most time-consuming job. ▪

Lemma 2. The optimal makespan $L^* \geq \frac{1}{m} \sum_k t_k$.

Pf.

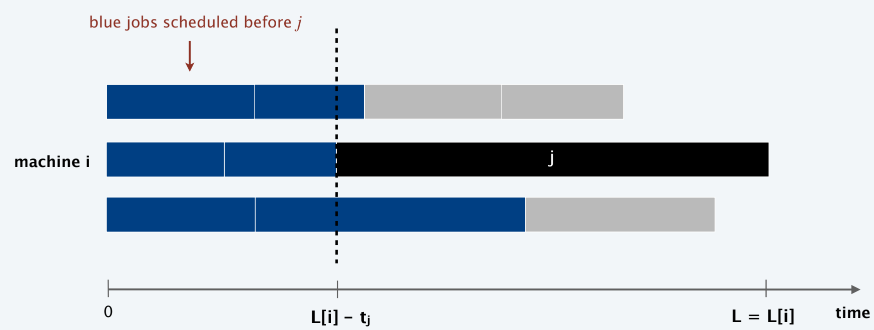
- The total processing time is $\Sigma_k\, t_k$.
- One of m machines must do at least a $1 / m$ fraction of total work. ▪

Load balancing: list scheduling analysis

Theorem. Greedy algorithm is a 2-approximation.

Pf. Consider load $L[i]$ of bottleneck machine i. ← machine that ends up with highest load

- Let j be last job scheduled on machine i.
- When job j assigned to machine i, i had smallest load. Its load before assignment is $L[i] - t_j$; hence $L[i] - t_j \leq L[k]$ for all $1 \leq k \leq m$.



Load balancing: list scheduling analysis

Theorem. Greedy algorithm is a 2-approximation.

Pf. Consider load $L[i]$ of bottleneck machine i. ← machine that ends up with highest load

- Let j be last job scheduled on machine i.
- When job j assigned to machine i, i had smallest load. Its load before assignment is $L[i] - t_j$; hence $L[i] - t_j \le L[k]$ for all $1 \le k \le m$.
- Sum inequalities over all k and divide by m:

$$
\begin{aligned}
L[i] - t_j &\le \frac{1}{m} \sum_k L[k] \\
&= \frac{1}{m} \sum_k t_k \\
&\le L^*. \quad \text{(Lemma 2)}
\end{aligned}
$$

- Now, $L = L[i] = \underbrace{(L[i] - t_j)}_{\le L^*} + \underbrace{t_j}_{\le L^*} \le 2L^*$ ▪

above inequality   Lemma 1

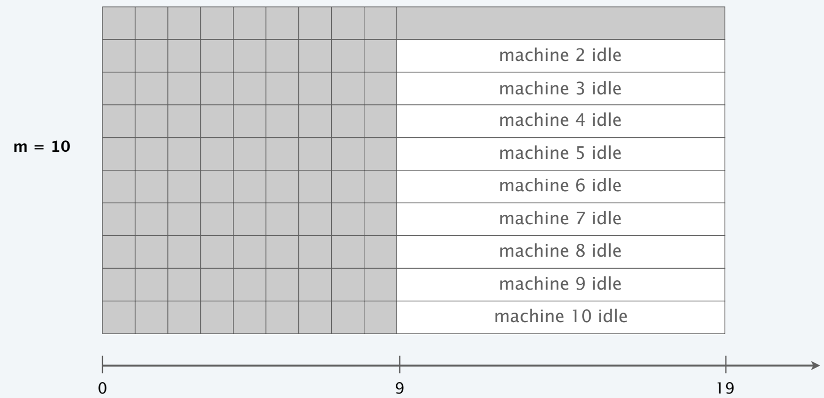
Load balancing: list scheduling analysis

Q. Is our analysis tight?

A. Essentially yes.

Ex: m machines, first $m(m-1)$ jobs have length 1, last job has length m.

list scheduling makespan = 19 = 2m − 1



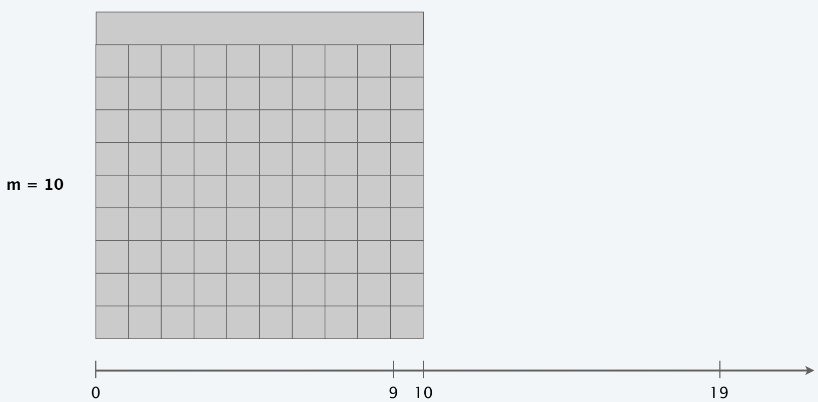
Load balancing: list scheduling analysis

Q. Is our analysis tight?

A. Essentially yes.

Ex: m machines, first $m(m-1)$ jobs have length 1, last job has length m.

optimal makespan = 10 = m



Load balancing: LPT rule

Longest processing time (LPT). Sort n jobs in decreasing order of processing times; then run list scheduling algorithm.

LPT-LIST-SCHEDULING $(m, n, t_1, t_2, \ldots, t_n)$

SORT jobs and renumber so that $t_1 \ge t_2 \ge \ldots \ge t_n$.

FOR $i = 1$ TO m
  $L[i] \leftarrow 0.$ ← load on machine i
  $S[i] \leftarrow \varnothing.$ ← jobs assigned to machine i

FOR $j = 1$ TO n
  $i \leftarrow \text{argmin}_k\, L[k].$ ← machine i has smallest load
  $S[i] \leftarrow S[i] \cup \{j\}.$ ← assign job j to machine i
  $L[i] \leftarrow L[i] + t_j.$ ← update load of machine i

RETURN $S[1], S[2], \ldots, S[m].$

Load balancing: LPT rule

Observation. If bottleneck machine i has only 1 job, then optimal.
Pf. Any solution must schedule that job. ▪

Lemma 3. If there are more than m jobs, $L^* \ge 2\, t_{m+1}$.
Pf.
- Consider processing times of first $m+1$ jobs $t_1 \ge t_2 \ge \ldots \ge t_{m+1}$.
- Each takes at least t_{m+1} time.
- There are $m+1$ jobs and m machines, so by pigeonhole principle, at least one machine gets two jobs. ▪

Theorem. LPT rule is a 3/2-approximation algorithm.
Pf. [similar to proof for list scheduling]
- Consider load $L[i]$ of bottleneck machine i.
- Let j be last job scheduled on machine i. assuming machine i has at least 2 jobs, we have $j \ge m+1$

$$L = L[i] = \underbrace{(L[i]-t_j)}_{\le L^*} + \underbrace{t_j}_{\le \frac{1}{2} L^*} \le \frac{3}{2} L^* \ \blacksquare$$

as before

Lemma 3 (since $t_{m+1} \ge t_j$)

Load balancing: LPT rule

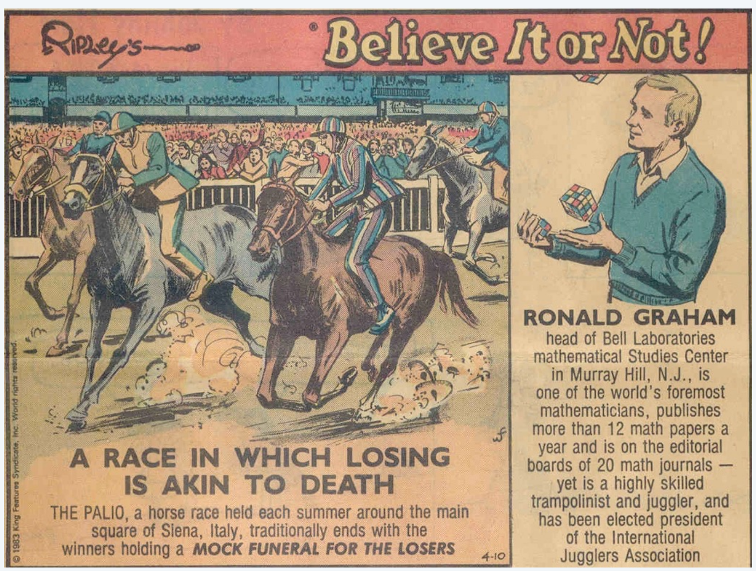
Q. Is our 3/2 analysis tight?
A. No.

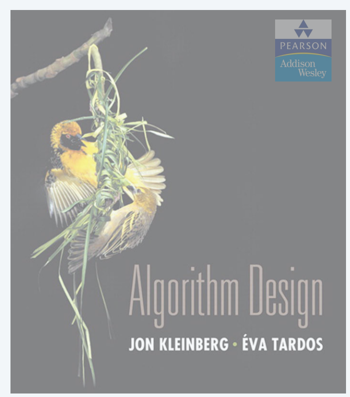
Theorem. [Graham 1969] LPT rule is a 4/3-approximation.
Pf. More sophisticated analysis of same algorithm.

Q. Is Graham's 4/3 analysis tight?
A. Essentially yes.

Ex.
- m machines
- $n = 2m + 1$ jobs
- 2 jobs of length $m, m+1, \ldots, 2m-1$ and one more job of length m.
- Then, $L / L^* = (4m-1) / (3m)$

Believe it or not





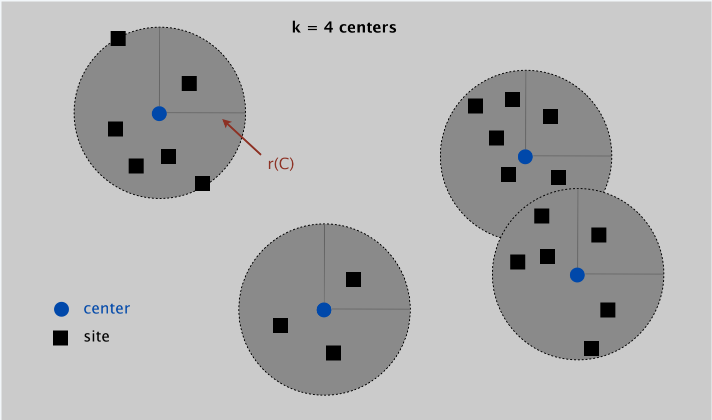
Section 11.2

11. Approximation Algorithms

Center selection problem

Input. Set of n sites $s_1, \ldots, s_n$ and an integer $k > 0$.

Center selection problem. Select set of k centers C so that maximum distance $r(C)$ from a site to nearest center is minimized.



Center selection problem

Input. Set of n sites $s_1, \ldots, s_n$ and an integer $k > 0$.

Center selection problem. Select set of k centers C so that maximum distance $r(C)$ from a site to nearest center is minimized.

Notation.

- $dist(x, y)$ = distance between sites x and y.
- $dist(s_i, C) = \min_{c \in C} dist(s_i, c)$ = distance from s_i to closest center.
- $r(C) = \max_i dist(s_i, C)$ = smallest covering radius.

Goal. Find set of centers C that minimizes $r(C)$, subject to $|C| = k$.

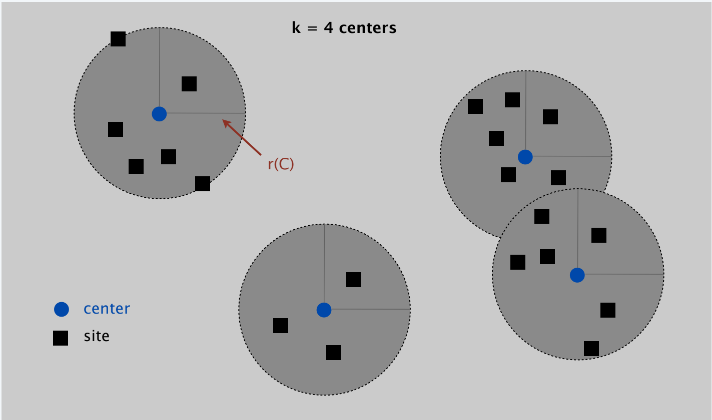
Distance function properties.

- $dist(x, x) = 0$ [identity]
- $dist(x, y) = dist(y, x)$ [symmetry]
- $dist(x, y) \leq dist(x, z) + dist(z, y)$ [triangle inequality]

Center selection example

Ex: each site is a point in the plane, a center can be any point in the plane, $dist(x, y)$ = Euclidean distance.

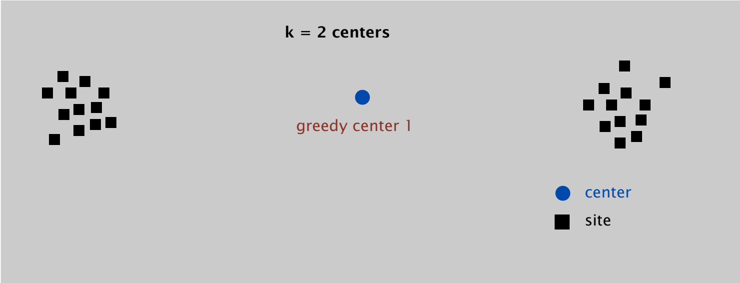
Remark: search can be infinite!



Greedy algorithm: a false start

Greedy algorithm. Put the first center at the best possible location for a single center, and then keep adding centers so as to reduce the covering radius each time by as much as possible.

Remark: arbitrarily bad!



Center selection: greedy algorithm

Repeatedly choose next center to be site farthest from any existing center.

GREEDY-CENTER-SELECTION $(k, n, s_1, s_2, \ldots, s_n)$

$C \leftarrow \varnothing$.

REPEAT k *times*

 Select a site s_i *with maximum distance* $dist(s_i, C)$.

 $C \leftarrow C \cup s_i$.

RETURN C.

site farthest from any center

Property. Upon termination, all centers in C are pairwise at least $r(C)$ apart.

Pf. By construction of algorithm.

Center selection: analysis of greedy algorithm

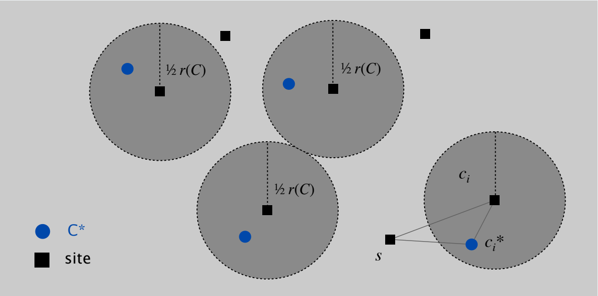
Lemma. Let C^* be an optimal set of centers. Then $r(C) \le 2r(C^*)$.

Pf. [by contradiction] Assume $r(C^*) < ½\, r(C)$.

- For each site $c_i \in C$, consider ball of radius $½\, r(C)$ around it.
- Exactly one c_i^* in each ball; let c_i be the site paired with c_i^*.
- Consider any site s and its closest center $c_i^* \in C^*$.
- $dist(s, C) \le dist(s, c_i) \le dist(s, c_i^*) + dist(c_i^*, c_i) \le 2r(C^*)$.
- Thus, $r(C) \le 2r(C^*)$. ▪

Δ-inequality

≤ r(C*) since c_i^* is closest center



Center selection

Lemma. Let C^* be an optimal set of centers. Then $r(C) \le 2r(C^*)$.

Theorem. Greedy algorithm is a 2-approximation for center selection problem.

Remark. Greedy algorithm always places centers at sites, but is still within a factor of 2 of best solution that is allowed to place centers anywhere.

e.g., points in the plane

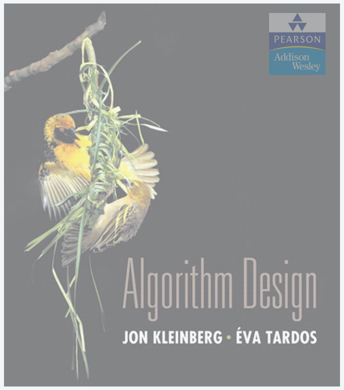
Question. Is there hope of a 3/2-approximation? 4/3?

Dominating set reduces to center selection

Theorem. Unless **P = NP**, there no ρ-approximation for center selection problem for any ρ < 2.

Pf. We show how we could use a $(2 - \varepsilon)$ approximation algorithm for CENTER-SELECTION selection to solve DOMINATING-SET in poly-time.

- Let $G = (V, E)$, k be an instance of DOMINATING-SET.
- Construct instance G' of CENTER-SELECTION with sites V and distances
  - $dist(u, v) = 1$ if $(u, v) \in E$
  - $dist(u, v) = 2$ if $(u, v) \notin E$
- Note that G' satisfies the triangle inequality.
- G has dominating set of size k iff there exists k centers C^* with $r(C^*) = 1$.
- Thus, if G has a dominating set of size k, a $(2 - \varepsilon)$-approximation algorithm for CENTER-SELECTION would find a solution C^* with $r(C^*) = 1$ since it cannot use any edge of distance 2. ▪



SECTION 11.4

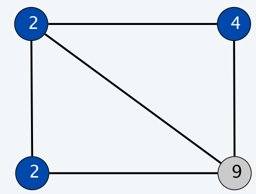
11. APPROXIMATION ALGORITHMS

- load balancing
- center selection
- *pricing method: weighted vertex cover*
- LP rounding: weighted vertex cover
- generalized load balancing
- knapsack problem

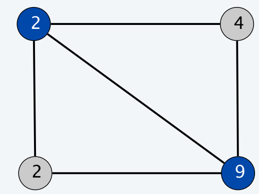
Weighted vertex cover

Definition. Given a graph $G = (V, E)$, a vertex cover is a set $S \subseteq V$ such that each edge in E has at least one end in S.

Weighted vertex cover. Given a graph G with vertex weights, find a vertex cover of minimum weight.



weight = 2 + 2 + 4



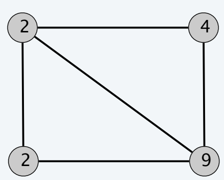
weight = 11

Pricing method

Pricing method. Each edge must be covered by some vertex.
Edge $e = (i, j)$ pays price $p_e \geq 0$ to use both vertex i and j.

Fairness. Edges incident to vertex i should pay $\leq w_i$ in total.

for each vertex i: $\sum_{e=(i,j)} p_e \leq w_i$



Fairness lemma. For any vertex cover S and any fair prices p_e: $\sum_e p_e \leq w(S)$.

Pf. $$\sum_{e \in E} p_e \leq \sum_{i \in S} \sum_{e=(i,j)} p_e \leq \sum_{i \in S} w_i = w(S). \blacksquare$$

each edge e covered by at least one node in S

sum fairness inequalities for each node in S

Pricing method

Set prices and find vertex cover simultaneously.

WEIGHTED-VERTEX-COVER (G, w)

$S \leftarrow \varnothing$.

FOREACH $e \in E$

  $p_e \leftarrow 0$.

$\sum_{e=(i,j)} p_e = w_i$

WHILE (there exists an edge (i, j) such that neither i nor j is tight)

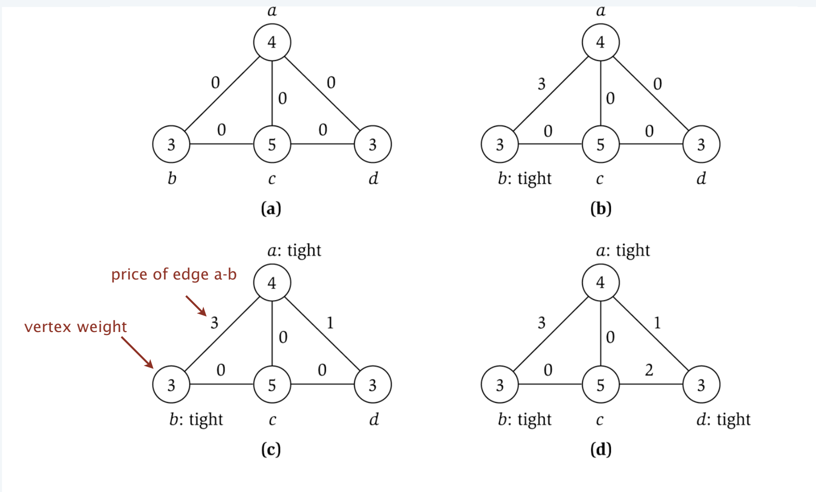
  Select such an edge $e = (i, j)$.

  Increase p_e as much as possible until i or j tight.

$S \leftarrow$ set of all tight nodes.

RETURN S.

Pricing method example



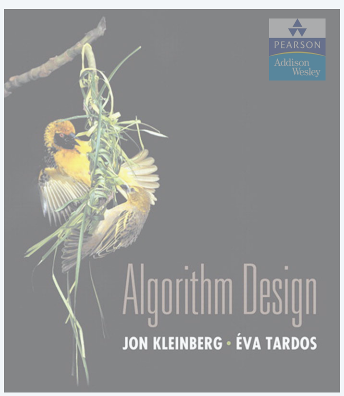
Pricing method: analysis

Theorem. Pricing method is a 2-approximation for WEIGHTED-VERTEX-COVER.

Pf.

- Algorithm terminates since at least one new node becomes tight after each iteration of while loop.
- Let S = set of all tight nodes upon termination of algorithm. S is a vertex cover: if some edge (i, j) is uncovered, then neither i nor j is tight. But then while loop would not terminate.
- Let S^* be optimal vertex cover. We show $w(S) \leq 2\, w(S^*)$.

$$w(S) = \sum_{i \in S} w_i \underset{\text{all nodes in S are tight}}{=} \sum_{i \in S} \sum_{e=(i,j)} p_e \underset{S \subseteq V,\ \text{prices} \geq 0}{\leq} \sum_{i \in V} \sum_{e=(i,j)} p_e \underset{\text{each edge counted twice}}{=} 2 \sum_{e \in E} p_e \underset{\text{fairness lemma}}{\leq} 2w(S^*). \quad \blacksquare$$



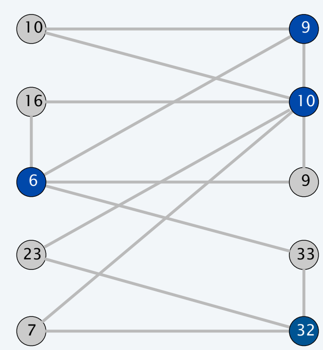
SECTION 11.6

11. APPROXIMATION ALGORITHMS

- load balancing
- center selection
- pricing method: weighted vertex cover
- **LP rounding: weighted vertex cover**
- generalized load balancing
- knapsack problem

Weighted vertex cover

Given a graph $G = (V, E)$ with vertex weights $w_i \geq 0$, find a min-weight subset of vertices $S \subseteq V$ such that every edge is incident to at least one vertex in S.



total weight = 6 + 9 + 10 + 32 = 57

Weighted vertex cover: ILP formulation

Given a graph $G = (V, E)$ with vertex weights $w_i \geq 0$, find a min-weight subset of vertices $S \subseteq V$ such that every edge is incident to at least one vertex in S.

Integer linear programming formulation.

- Model inclusion of each vertex i using a 0/1 variable x_i.

$$x_i = \begin{cases} 0 & \text{if vertex } i \text{ is not in vertex cover} \\ 1 & \text{if vertex } i \text{ is in vertex cover} \end{cases}$$

  Vertex covers in 1–1 correspondence with 0/1 assignments: $S = \{ i \in V : x_i = 1\}$.

- Objective function: minimize $\Sigma_i w_i x_i$.

- For every edge (i, j), must take either vertex i or j (or both): $x_i + x_j \geq 1$.

Weighted vertex cover: ILP formulation

Weighted vertex cover. Integer linear programming formulation.

$$\begin{array}{llrcll} (ILP) & \min & \sum_{i \in V} w_i \, x_i & & & \\ & \text{s.t.} & x_i + x_j & \geq & 1 & (i,j) \in E \\ & & x_i & \in & \{0,\ 1\} & i \in V \end{array}$$

Observation. If x^* is optimal solution to ILP, then $S = \{ i \in V : x_i^* = 1\}$ is a min-weight vertex cover.

Integer linear programming

Given integers a_{ij}, b_i, and c_j, find integers x_j that satisfy:

$$\begin{array}{lrcl} \min & c^{\mathsf{T}} x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \\ & x & & \text{integral} \end{array}$$

$$\begin{array}{lrcll} \min & \sum_{j=1}^{n} c_j x_j & & & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & \geq & b_i & 1 \leq i \leq m \\ & x_j & \geq & 0 & 1 \leq j \leq n \\ & x_j & & \text{integral} & 1 \leq j \leq n \end{array}$$

Observation. Vertex cover formulation proves that INTEGER-PROGRAMMING is an **NP**-hard optimization problem.

Linear programming

Given integers a_{ij}, b_i, and c_j, find real numbers x_j that satisfy:

$$\begin{array}{lrcl} \min & c^{\mathsf{T}} x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

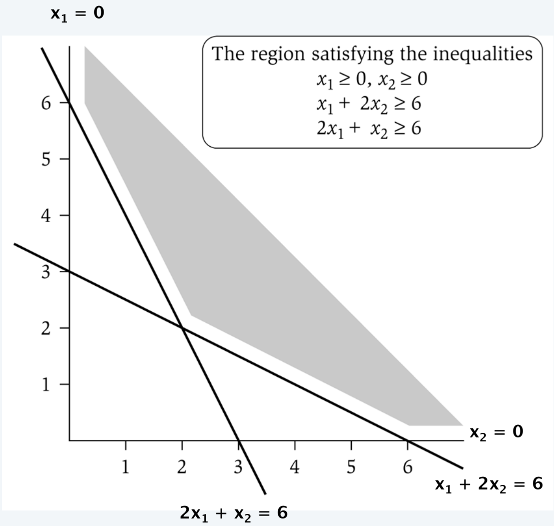
$$\begin{array}{lrcll} \min & \sum_{j=1}^{n} c_j x_j & & & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & \geq & b_i & 1 \leq i \leq m \\ & x_j & \geq & 0 & 1 \leq j \leq n \end{array}$$

Linear. No x^2, xy, $\arccos(x)$, $x(1-x)$, etc.

Simplex algorithm. [Dantzig 1947] Can solve LP in practice.
Ellipsoid algorithm. [Khachiyan 1979] Can solve LP in poly-time.
Interior point algorithms. [Karmarkar 1984, Renegar 1988, …]
Can solve LP both in poly-time and in practice.

LP feasible region

LP geometry in 2D.



Weighted vertex cover: LP relaxation

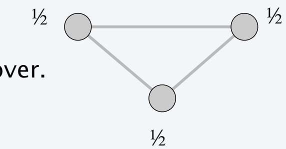
Linear programming relaxation.

$$
\begin{array}{llrcll}
(LP) & \min & \sum_{i \in V} w_i\, x_i & & & \\
 & \text{s.t.} & x_i + x_j & \ge & 1 & (i,j) \in E \\
 & & x_i & \ge & 0 & i \in V
\end{array}
$$

Observation. Optimal value of LP is $\le$ optimal value of ILP.

Pf. LP has fewer constraints.

Note. LP solution x^* may not correspond to a vertex cover.
(even if all weights are 1)



Q. How can solving LP help us find a low-weight vertex cover?

A. Solve LP and round fractional values in x^*.

Weighted vertex cover: LP rounding algorithm

Lemma. If x^* is optimal solution to LP, then $S = \{ i \in V : x_i^* \ge ½ \}$ is a vertex cover whose weight is at most twice the min possible weight.

Pf. [S is a vertex cover]

- Consider an edge $(i, j) \in E$.
- Since $x_i^* + x_j^* \ge 1$, either $x_i^* \ge ½$ or $x_j^* \ge ½$ (or both) $\Rightarrow$ (i, j) covered.

Pf. [S has desired weight]

- Let S^* be optimal vertex cover. Then

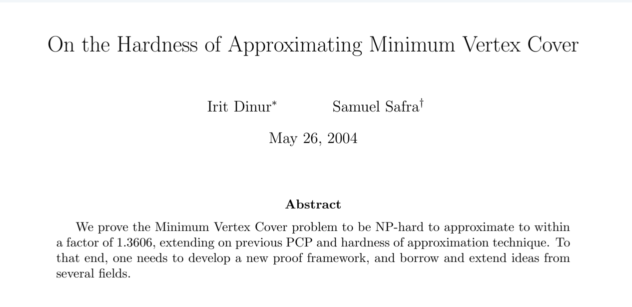
$$
\sum_{i \in S^*} w_i \;\underset{\substack{\uparrow \\ \text{LP is a relaxation}}}{\ge}\; \sum_{i \in S} w_i\, x_i^* \;\underset{\substack{\uparrow \\ x_i^* \,\ge\, ½}}{\ge}\; \frac{1}{2} \sum_{i \in S} w_i
$$

Theorem. The rounding algorithm is a 2-approximation algorithm.

Pf. Lemma + fact that LP can be solved in poly-time.

Weighted vertex cover inapproximability

Theorem. [Dinur–Safra 2004] If **P ≠ NP**, then no ρ-approximation algorithm for WEIGHTED-VERTEX-COVER for any $\rho < 1.3606$ (even if all weights are 1).



On the Hardness of Approximating Minimum Vertex Cover

Irit Dinur* Samuel Safra†

May 26, 2004

Abstract

We prove the Minimum Vertex Cover problem to be NP-hard to approximate to within a factor of 1.3606, extending on previous PCP and hardness of approximation technique. To that end, one needs to develop a new proof framework, and borrow and extend ideas from several fields.

Open research problem. Close the gap.

Weighted vertex cover inapproximability

Theorem. [Kohot–Regev 2008] If Unique Games Conjecture is true, then no $2 - \varepsilon$ approximation algorithm for WEIGHTED-VERTEX-COVER for any $\varepsilon > 0$.

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JOURNAL OF COMPUTER AND SYSTEM SCIENCES

Journal of Computer and System Sciences 74 (2008) 335–349

Vertex cover might be hard to approximate to within $2 - \varepsilon$

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Received 28 May 2003; received in revised form 25 April 2006

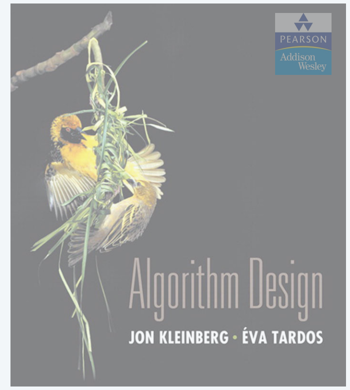
Available online 13 June 2007

Abstract

Based on a conjecture regarding the power of unique 2-prover-1-round games presented in [S. Khot, On the power of unique 2-Prover 1-Round games, in: Proc. 34th ACM Symp. on Theory of Computing, STOC, May 2002, pp. 767–775], we show that vertex cover is hard to approximate within any constant factor better than 2. We actually show a stronger result, namely, based on the same conjecture, vertex cover on k-uniform hypergraphs is hard to approximate within any constant factor better than k.

Keywords: Hardness of approximation; Vertex cover; Unique games conjecture

Open research problem. Prove the Unique Games Conjecture.



SECTION 11.7

11. APPROXIMATION ALGORITHMS

- load balancing
- center selection
- pricing method: weighted vertex cover
- LP rounding: weighted vertex cover
- *generalized load balancing*
- knapsack problem

Generalized load balancing

Input. Set of m machines M; set of n jobs J.

- Job $j \in J$ must run contiguously on an authorized machine in $M_j \subseteq M$.
- Job $j \in J$ has processing time t_j.
- Each machine can process at most one job at a time.

Def. Let J_i be the subset of jobs assigned to machine i.
The load of machine i is $L_i = \Sigma_{j \in J_i} t_j$.

Def. The makespan is the maximum load on any machine $= \max_i L_i$.

Generalized load balancing. Assign each job to an authorized machine to minimize makespan.

Generalized load balancing: integer linear program and relaxation

ILP formulation. x_{ij} = time machine i spends processing job j.

$$
\begin{array}{llllll}
(IP) & \min & L & & & \\
& \text{s. t.} & \sum_i x_{ij} & = & t_j & \text{for all } j \in J \\
& & \sum_j x_{ij} & \leq & L & \text{for all } i \in M \\
& & x_{ij} & \in & \{0, t_j\} & \text{for all } j \in J \text{ and } i \in M_j \\
& & x_{ij} & = & 0 & \text{for all } j \in J \text{ and } i \notin M_j
\end{array}
$$

LP relaxation.

$$
\begin{array}{llllll}
(LP) & \min & L & & & \\
& \text{s. t.} & \sum_i x_{ij} & = & t_j & \text{for all } j \in J \\
& & \sum_j x_{ij} & \leq & L & \text{for all } i \in M \\
& & x_{ij} & \geq & 0 & \text{for all } j \in J \text{ and } i \in M_j \\
& & x_{ij} & = & 0 & \text{for all } j \in J \text{ and } i \notin M_j
\end{array}
$$

Generalized load balancing: lower bounds

Lemma 1. The optimal makespan $L^* \geq \max_j t_j$.

Pf. Some machine must process the most time-consuming job. ▪

Lemma 2. Let L be optimal value to the LP. Then, optimal makespan $L^* \geq L$.

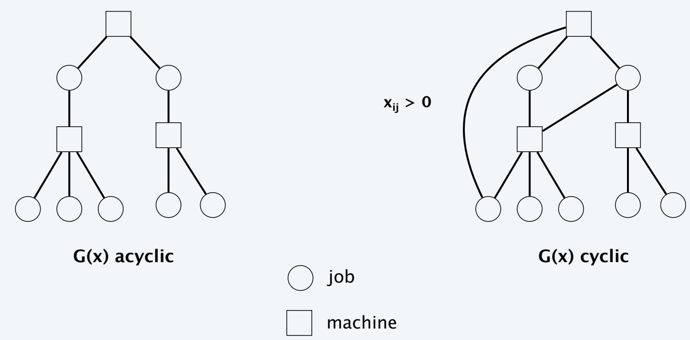
Pf. LP has fewer constraints than ILP formulation. ▪

Generalized load balancing: structure of LP solution

Lemma 3. Let x be solution to LP. Let $G(x)$ be the graph with an edge between machine i and job j if $x_{ij} > 0$. Then $G(x)$ is acyclic.

can transform x into another LP solution where G(x) is acyclic if LP solver doesn't return such an x

Pf. (deferred)



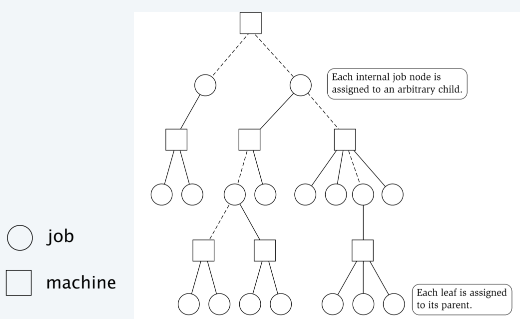
Generalized load balancing: rounding

Rounded solution. Find LP solution x where $G(x)$ is a forest. Root forest $G(x)$ at some arbitrary machine node r.

- If job j is a leaf node, assign j to its parent machine i.
- If job j is not a leaf node, assign j to any one of its children.

Lemma 4. Rounded solution only assigns jobs to authorized machines.

Pf. If job j is assigned to machine i, then $x_{ij} > 0$. LP solution can only assign positive value to authorized machines. ▪



Generalized load balancing: analysis

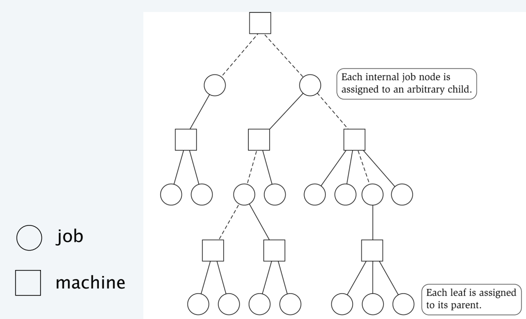
Lemma 5. If job j is a leaf node and machine $i = parent(j)$, then $x_{ij} = t_j$.

Pf.

- Since i is a leaf, $x_{ij} = 0$ for all $j \neq parent(i)$.
- LP constraint guarantees $\Sigma_i\, x_{ij} = t_j$. ▪

Lemma 6. At most one non-leaf job is assigned to a machine.

Pf. The only possible non-leaf job assigned to machine i is $parent(i)$. ▪



Generalized load balancing: analysis

Theorem. Rounded solution is a 2-approximation.

Pf.

- Let $J(i)$ be the jobs assigned to machine i.
- By LEMMA 6, the load L_i on machine i has two components:
  - leaf nodes:

$$\sum_{\substack{j \in J(i) \\ j \text{ is a leaf}}} t_j \overset{\text{Lemma 5}}{=} \sum_{\substack{j \in J(i) \\ j \text{ is a leaf}}} x_{ij} \le \sum_{j \in J} x_{ij} \overset{\text{LP}}{\le} \underbrace{L}_{\text{optimal value of LP}} \overset{\text{Lemma 2 (LP is a relaxation)}}{\le} L^*$$

  - parent:

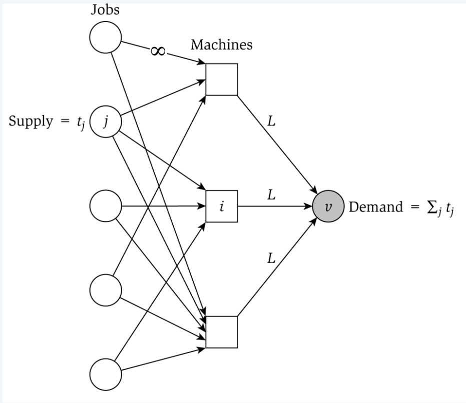
$$t_{\text{parent}(i)} \overset{\text{Lemma 1}}{\le} L^*$$

- Thus, the overall load $L_i \le 2L^*$. ▪

Generalized load balancing: flow formulation

Flow formulation of LP.

$$\begin{array}{rcll} \sum_i x_{ij} & = & t_j & \text{for all } j \in J \\ \sum_j x_{ij} & \le & L & \text{for all } i \in M \\ x_{ij} & \ge & 0 & \text{for all } j \in J \text{ and } i \in M_j \\ x_{ij} & = & 0 & \text{for all } j \in J \text{ and } i \notin M_j \end{array}$$



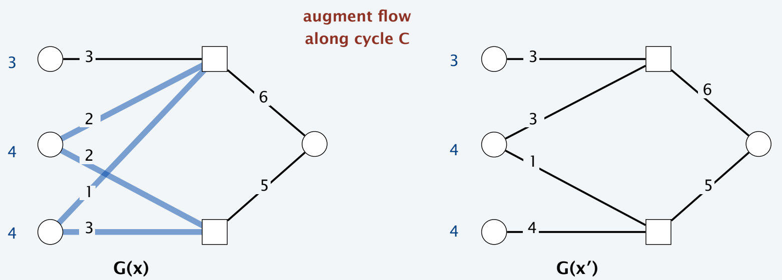
Observation. Solution to feasible flow problem with value L are in 1-to-1 correspondence with LP solutions of value L.

Generalized load balancing: structure of solution

Lemma 3. Let (x, L) be solution to LP. Let $G(x)$ be the graph with an edge from machine i to job j if $x_{ij} > 0$. We can find another solution (x', L) such that $G(x')$ is acyclic.

Pf. Let C be a cycle in $G(x)$.

- Augment flow along the cycle C. ← flow conservation maintained
- At least one edge from C is removed (and none are added).
- Repeat until $G(x')$ is acyclic. ▪



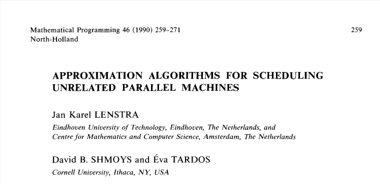
Conclusions

Running time. The bottleneck operation in our 2-approximation is solving one LP with $mn + 1$ variables.

Remark. Can solve LP using flow techniques on a graph with $m + n + 1$ nodes: given L, find feasible flow if it exists. Binary search to find L^*.

Extensions: unrelated parallel machines. [Lenstra–Shmoys–Tardos 1990]

- Job j takes t_{ij} time if processed on machine i.
- 2-approximation algorithm via LP rounding.
- If **P** ≠ **NP**, then no no ρ-approximation exists for any $\rho < 3/2$.


Mathematical Programming 46 (1990) 259-271
North-Holland 259

APPROXIMATION ALGORITHMS FOR SCHEDULING UNRELATED PARALLEL MACHINES

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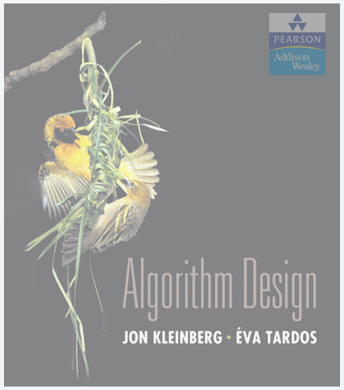

SECTION 11.8

11. APPROXIMATION ALGORITHMS

- load balancing
- center selection
- pricing method: weighted vertex cover
- LP rounding: weighted vertex cover
- generalized load balancing
- *knapsack problem*

Polynomial-time approximation scheme

PTAS. $(1 + \varepsilon)$-approximation algorithm for any constant $\varepsilon > 0$.

- Load balancing. [Hochbaum–Shmoys 1987]
- Euclidean TSP. [Arora, Mitchell 1996]

Consequence. PTAS produces arbitrarily high quality solution, but trades off accuracy for time.

This section. PTAS for knapsack problem via rounding and scaling.

Knapsack problem

Knapsack problem.

- Given n objects and a knapsack.
- Item i has value $v_i > 0$ and weighs $w_i > 0$. ← we assume $w_i \le W$ for each i
- Knapsack has weight limit W.
- Goal: fill knapsack so as to maximize total value.

Ex: $\{ 3, 4 \}$ has value 40.

item	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

original instance (W = 11)

Knapsack is NP-complete

KNAPSACK. Given a set X, weights $w_i \ge 0$, values $v_i \ge 0$, a weight limit W, and a target value V, is there a subset $S \subseteq X$ such that:

$$\sum_{i \in S} w_i \le W$$
$$\sum_{i \in S} v_i \ge V$$

SUBSET-SUM. Given a set X, values $u_i \ge 0$, and an integer U, is there a subset $S \subseteq X$ whose elements sum to exactly U?

Theorem. SUBSET-SUM $\le_P$ KNAPSACK.

Pf. Given instance $(u_1, \ldots, u_n, U)$ of SUBSET-SUM, create KNAPSACK instance:

$$v_i = w_i = u_i \qquad \sum_{i \in S} u_i \le U$$
$$V = W = U \qquad \sum_{i \in S} u_i \ge U$$

Knapsack problem: dynamic programming I

Def. $OPT(i, w)$ = max value subset of items $1,..., i$ with weight limit w.

Case 1. OPT does not select item i.

- OPT selects best of $1, …, i – 1$ using up to weight limit w.

Case 2. OPT selects item i.

- New weight limit $= w – w_i$.
- OPT selects best of $1, …, i – 1$ using up to weight limit $w – w_i$.

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{\, OPT(i-1,w),\quad v_i + OPT(i-1, w-w_i) \,\} & \text{otherwise} \end{cases}$$

Theorem. Computes the optimal value in $O(n\,W)$ time.

- Not polynomial in input size.
- Polynomial in input size if weights are small integers.

Knapsack problem: dynamic programming II

Def. $OPT(i, v)$ = min weight of a knapsack for which we can obtain a solution of value $\geq v$ using a subset of items $1,..., i$.

Note. Optimal value is the largest value v such that $OPT(n, v) \leq W$.

Case 1. OPT does not select item i.

- OPT selects best of $1, …, i – 1$ that achieves value $\geq v$.

Case 2. OPT selects item i.

- Consumes weight w_i, need to achieve value $\geq v – v_i$.
- OPT selects best of $1, …, i – 1$ that achieves value $\geq v – v_i$.

$$OPT(i,v) = \begin{cases} 0 & \text{if } v \leq 0 \\ \infty & \text{if } i = 0 \text{ and } v > 0 \\ \min\{OPT(i-1,v),\ w_i + OPT(i-1, v-v_i)\} & \text{otherwise} \end{cases}$$

Knapsack problem: dynamic programming II

Theorem. Dynamic programming algorithm II computes the optimal value in $O(n^2\, v_{max})$ time, where v_{max} is the maximum of any value.

Pf.

- The optimal value $V^* \leq n\, v_{max}$.
- There is one subproblem for each item and for each value $v \leq V^*$.
- It takes $O(1)$ time per subproblem. ▪

Remark 1. Not polynomial in input size!

Remark 2. Polynomial time if values are small integers.

Knapsack problem: polynomial-time approximation scheme

Intuition for approximation algorithm.

- Round all values up to lie in smaller range.
- Run dynamic programming algorithm II on rounded/scaled instance.
- Return optimal items in rounded instance.

item	value	weight
1	934221	1
2	5956342	2
3	17810013	5
4	21217800	6
5	27343199	7

original instance (W = 11)

item	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

rounded instance (W = 11)

Knapsack problem: polynomial-time approximation scheme

Round up all values:

- $0 < \varepsilon \leq 1$ = precision parameter.
- v_{max} = largest value in original instance.
- θ = scaling factor = $\varepsilon \, v_{max} / 2n$.

$$\bar{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil \theta, \quad \hat{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil$$

Observation. Optimal solutions to problem with $\bar{v}$ are equivalent to optimal solutions to problem with $\hat{v}$.

Intuition. $\bar{v}$ close to v so optimal solution using $\bar{v}$ is nearly optimal; $\hat{v}$ small and integral so dynamic programming algorithm II is fast.

Knapsack problem: polynomial-time approximation scheme

Theorem. If S is solution found by rounding algorithm and S^* is any other feasible solution, then $(1+\epsilon)\sum_{i \in S} v_i \geq \sum_{i \in S^*} v_i$

Pf. Let S^* be any feasible solution satisfying weight constraint.

$$\begin{aligned}
\sum_{i \in S^*} v_i &\leq \sum_{i \in S^*} \bar{v}_i && \text{always round up} \\
&\leq \sum_{i \in S} \bar{v}_i && \text{solve rounded instance optimally} \\
&\leq \sum_{i \in S} (v_i + \theta) && \text{never round up by more than } \theta \\
&\leq \sum_{i \in S} v_i + n\theta && |S| \leq n \\
&= \sum_{i \in S} v_i + \tfrac{1}{2}\epsilon\, v_{\max} && \theta = \varepsilon\, v_{max} / 2n \\
&\leq (1+\epsilon)\sum_{i \in S} v_i && v_{max} \leq 2 \textstyle\sum_{i \in S} v_i
\end{aligned}$$

subset containing only the item of largest value

choosing $S^* = \{ \max \}$

$$v_{\max} \leq \sum_{i \in S} v_i + \tfrac{1}{2}\epsilon\, v_{\max}$$

$$\leq \sum_{i \in S} v_i + \tfrac{1}{2} v_{\max}$$

thus

$$v_{\max} \leq 2\sum_{i \in S} v_i$$

Knapsack problem: polynomial-time approximation scheme

Theorem. For any $\varepsilon > 0$, the rounding algorithm computes a feasible solution whose value is within a $(1 + \varepsilon)$ factor of the optimum in $O(n^3 / \varepsilon)$ time.

Pf.

- We have already proved the accuracy bound.
- Dynamic program II running time is $O(n^2 \, \hat{v}_{max})$, where

$$\hat{v}_{\max} = \left\lceil \frac{v_{\max}}{\theta} \right\rceil = \left\lceil \frac{2n}{\epsilon} \right\rceil$$