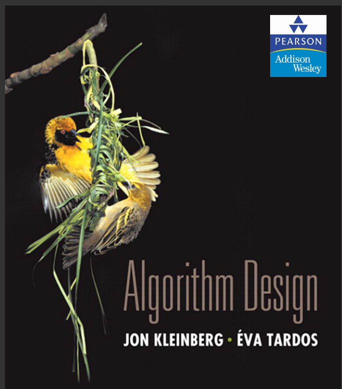


7. Network Flow II

- *bipartite matching*
- *disjoint paths*
- *extensions to max flow*
- *survey design*
- *airline scheduling*
- *image segmentation*
- *project selection*
- *baseball elimination*

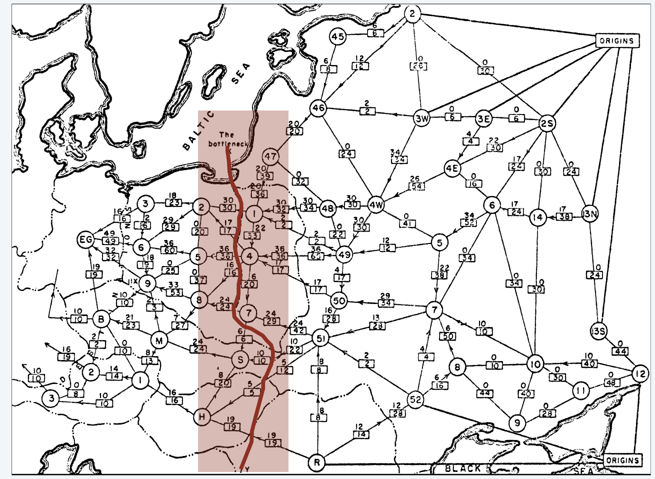
Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on 1/14/20 2:20 PM

Minimum cut application (RAND 1950s)

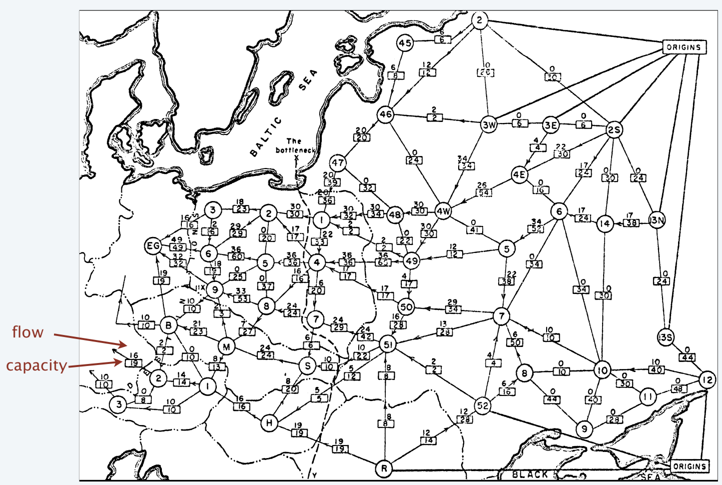
"Free world" goal. Cut supplies (if Cold War turns into real war).



rail network connecting Soviet Union with Eastern European countries
(map declassified by Pentagon in 1999)

Maximum flow application (Tolstoǐ 1930s)

Soviet Union goal. Maximize flow of supplies to Eastern Europe.



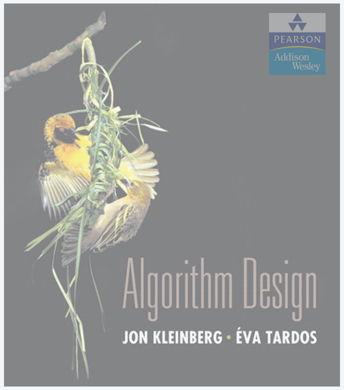
rail network connecting Soviet Union with Eastern European countries
(map declassified by Pentagon in 1999)

Max-flow and min-cut applications

Max-flow and min-cut problems are widely applicable model.

- Data mining.
- Open-pit mining.
- Bipartite matching.
- Network reliability.
- Baseball elimination.
- Image segmentation.
- Network connectivity.
- Markov random fields.
- Distributed computing.
- Security of statistical data.
- Egalitarian stable matching.
- Network intrusion detection.
- Multi-camera scene reconstruction.
- Sensor placement for homeland security.
- Many, many, more.

liver and hepatic vascularization segmentation



SECTION 7.5

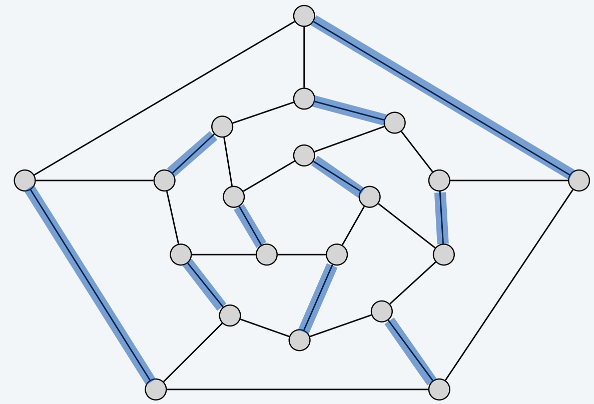
7. NETWORK FLOW II

- *bipartite matching*
- *disjoint paths*
- *extensions to max flow*
- *survey design*
- *airline scheduling*
- *image segmentation*
- *project selection*
- *baseball elimination*

Matching

Def. Given an undirected graph $G = (V, E)$, subset of edges $M \subseteq E$ is a matching if each node appears in at most one edge in M.

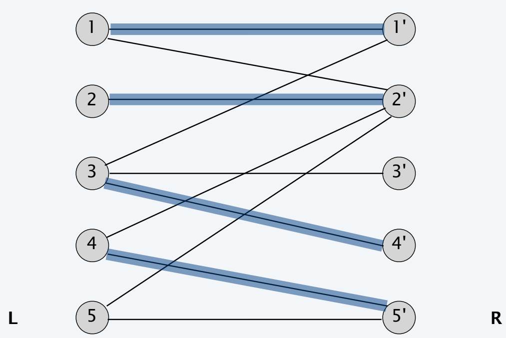
Max matching. Given a graph G, find a max-cardinality matching.



Bipartite matching

Def. A graph G is bipartite if the nodes can be partitioned into two subsets L and R such that every edge connects a node in L with a node in R.

Bipartite matching. Given a bipartite graph $G = (L \cup R, E)$, find a max-cardinality matching.

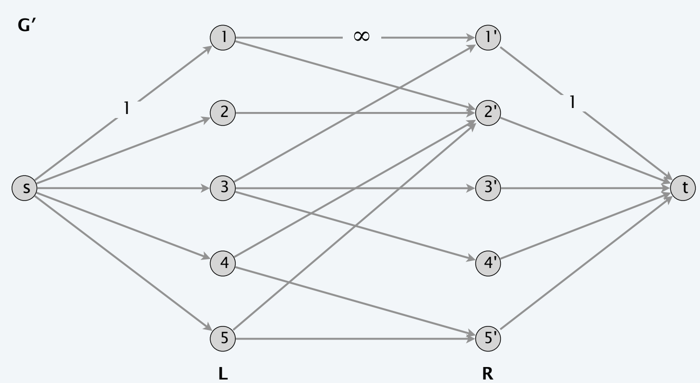


matching: 1–1', 2–2', 3–4', 4–5'

Bipartite matching: max-flow formulation

Formulation.

- Create digraph $G' = (L \cup R \cup \{s, t\},\ E'\ )$.
- Direct all edges from L to R, and assign infinite (or unit) capacity.
- Add unit-capacity edges from s to each node in L.
- Add unit-capacity edges from each node in R to t.



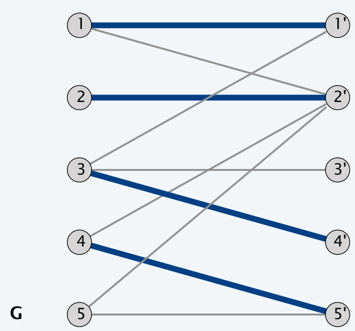
Max-flow formulation: proof of correctness

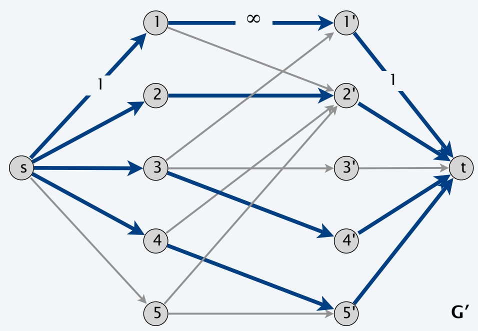
Theorem. 1–1 correspondence between matchings of cardinality k in G and integral flows of value k in G'.

for each edge e: $f(e) \in \{ 0, 1 \}$

Pf. $\Rightarrow$

- Let M be a matching in G of cardinality k.
- Consider flow f that sends 1 unit on each of the k corresponding paths.
- f is a flow of value k. ▪





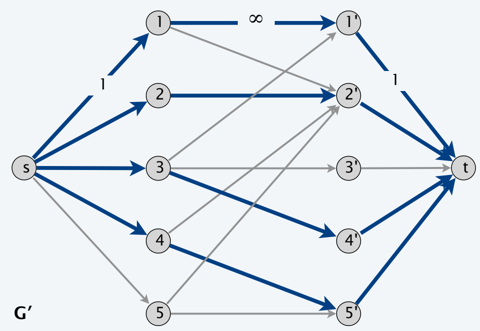
Max-flow formulation: proof of correctness

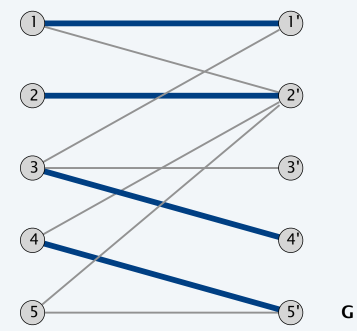
Theorem. 1–1 correspondence between matchings of cardinality k in G and integral flows of value k in G'.

for each edge e: $f(e) \in \{ 0, 1 \}$

Pf. $\Leftarrow$

- Let f be an integral flow in G' of value k.
- Consider M = set of edges from L to R with $f(e) = 1$.
  - each node in L and R participates in at most one edge in M
  - $|M| = k$: apply flow-value lemma to cut $(L \cup \{s\}, R \cup \{t\})$ ▪





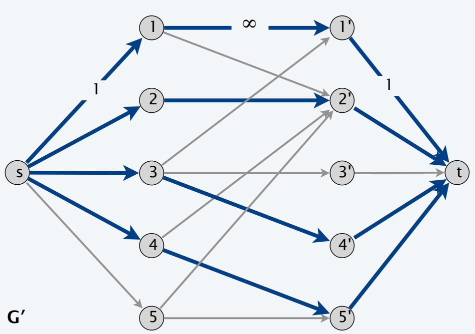
Max-flow formulation: proof of correctness

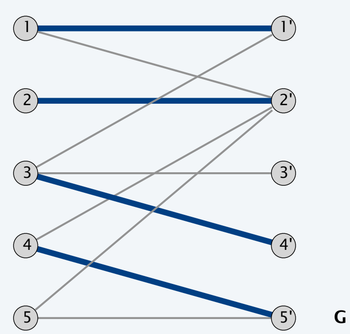
Theorem. 1–1 correspondence between matchings of cardinality k in G and integral flows of value k in G'.

Corollary. Can solve bipartite matching problem via max-flow formulation.

Pf.

- Integrality theorem $\Rightarrow$ there exists a max flow f^* in G' that is integral.
- 1–1 correspondence $\Rightarrow$ f^* corresponds to max-cardinality matching. ▪





Network flow II: quiz 1

What is running time of Ford–Fulkerson algorithms to find a max-cardinality matching in a bipartite graph with $|L| = |R| = n$?

A. $O(m + n)$

B. $O(mn)$

C. $O(mn^2)$

D. $O(m^2 n)$

Perfect matchings in bipartite graphs

Def. Given a graph $G = (V, E)$, a subset of edges $M \subseteq E$ is a perfect matching if each node appears in exactly one edge in M.

Q. When does a bipartite graph have a perfect matching?

Structure of bipartite graphs with perfect matchings.

- Clearly, we must have $|L| = |R|$.
- Which other conditions are necessary?
- Which other conditions are sufficient?

Perfect matchings in bipartite graphs

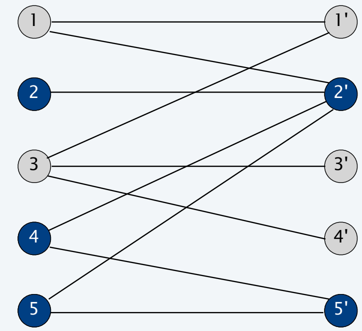
Notation. Let S be a subset of nodes, and let $N(S)$ be the set of nodes adjacent to nodes in S.

Observation. If a bipartite graph $G = (L \cup R, E)$ has a perfect matching, then $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

Pf. Each node in S has to be matched to a different node in $N(S)$. ▪

S = { 2, 4, 5 }

N(S) = { 2', 5' }



no perfect matching

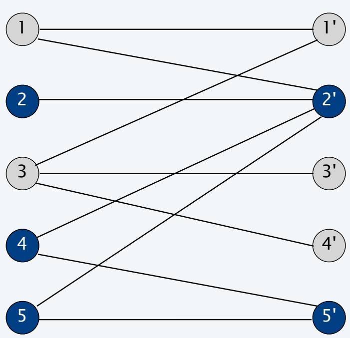
Hall's marriage theorem

Theorem. [Frobenius 1917, Hall 1935] Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then, graph G has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

Pf. $\Rightarrow$ This was the previous observation.

S = { 2, 4, 5 }

N(S) = { 2', 5' }

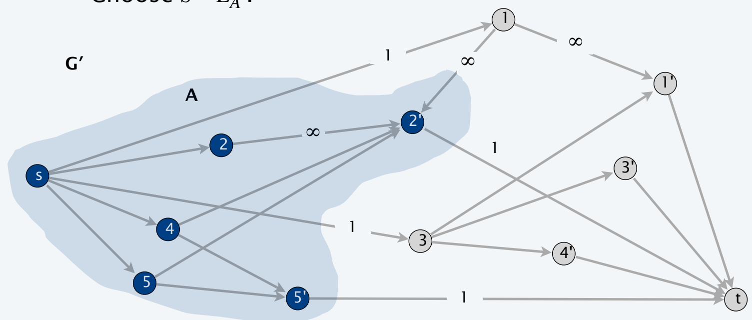


no perfect matching

Hall's marriage theorem

Pf. $\Leftarrow$ Suppose G does not have a perfect matching.

- Formulate as a max-flow problem and let (A, B) be a min cut in G'.
- By max-flow min-cut theorem, $cap(A, B) < |L|$.
- Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$.
- $cap(A, B) = |L_B| + |R_A| \Rightarrow |R_A| < |L_A|$.
- Min cut can't use ∞ edges $\Rightarrow N(L_A) \subseteq R_A$.
- $|N(L_A)| \leq |R_A| < |L_A|$.
- Choose $S = L_A$. ▪



$L_A = \{2, 4, 5\}$

$L_B = \{1, 3\}$

$R_A = \{2', 5'\}$

$N(L_A) = \{2', 5'\}$

Bipartite matching

Problem. Given a bipartite graph, find a max-cardinality matching.

year	worst case	technique	discovered by
1955	$O(m\ n)$	**augmenting path**	Ford–Fulkerson
1973	$O(m\ n^{1/2})$	**blocking flow**	Hopcroft–Karp, Karzanov
2004	$O(n^{2.378})$	**fast matrix multiplication**	Mucha–Sankowsi
2013	$\tilde{O}(m^{10/7})$	**electrical flow**	Mądry
20xx	???		

running time for finding a max-cardinality matching in a bipartite graph with n nodes and m edges

Network flow II: quiz 2

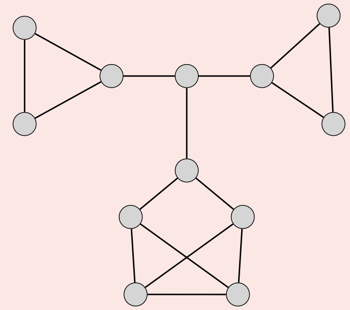
Which of the following are properties of the graph G = (V, E)?

A. G has a perfect matching.

B. Hall's condition is satisfied: $|N(S)| \geq |S|$ for all subsets $S \subseteq V$.

C. Both A and B.

D. Neither A nor B.



Nonbipartite matching

Problem. Given an undirected graph, find a max-cardinality matching.

- Structure of nonbipartite graphs is more complicated.
- But well understood. [Tutte–Berge formula, Edmonds–Gallai]
- Blossom algorithm: $O(n^4)$. [Edmonds 1965]
- Best known: $O(m\ n^{1/2})$. [Micali–Vazirani 1980, Vazirani 1994]

PATHS, TREES, AND FLOWERS

JACK EDMONDS

1. Introduction. A *graph G* for purposes here is a finite set of elements called *vertices* and a finite set of elements called *edges* such that each edge *meets* exactly two vertices, called the *end-points* of the edge. An edge is said to *join* its end-points.

A *matching* in G is a subset of its edges such that no two meet the same vertex. We describe an efficient algorithm for finding in a given graph a matching of maximum cardinality. This problem was posed and partly solved by C. Berge; see Sections 3.7 and 3.8.

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COMBINATORICA **14** (1) (1994) 71–109

A THEORY OF ALTERNATING PATHS AND BLOSSOMS FOR PROVING CORRECTNESS OF THE $O(\sqrt{V}E)$ GENERAL GRAPH MAXIMUM MATCHING ALGORITHM

VIJAY V. VAZIRANI[1]

Received December 30, 1989
Revised June 15, 1993

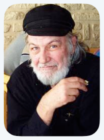
Historical significance (Jack Edmonds 1965)

2. Digression. An explanation is due on the use of the words "efficient algorithm." First, what I present is a conceptual description of an algorithm and not a particular formalized algorithm or "code."

For practical purposes computational details are vital. However, my purpose is only to show as attractively as I can that there is an efficient algorithm. According to the dictionary, "efficient" means "adequate in operation or performance." This is roughly the meaning I want—in the sense that it is conceivable for maximum matching to have no efficient algorithm. Perhaps a better word is "good."

I am claiming, as a mathematical result, the existence of a *good* algorithm for finding a maximum cardinality matching in a graph.

There is an obvious finite algorithm, but that algorithm increases in difficulty exponentially with the size of the graph. It is by no means obvious whether *or not* there exists an algorithm whose difficulty increases only algebraically with the size of the graph.



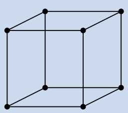
Hackathon problem

Hackathon problem.

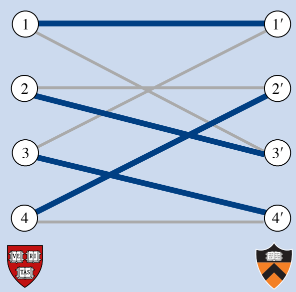
- Hackathon attended by n Harvard students and n Princeton students.
- Each Harvard student is friends with exactly $k > 0$ Princeton students; each Princeton student is friends with exactly k Harvard students.
- Is it possible to arrange the hackathon so that each Princeton student pair programs with a different friend from Harvard?

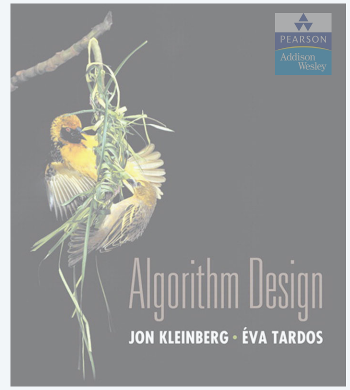
Mathematical reformulation. Does every k-regular bipartite graph have a perfect matching?

Ex. Boolean hypercube.



2-regular bipartite graph





Section 7.6

7. Network Flow II

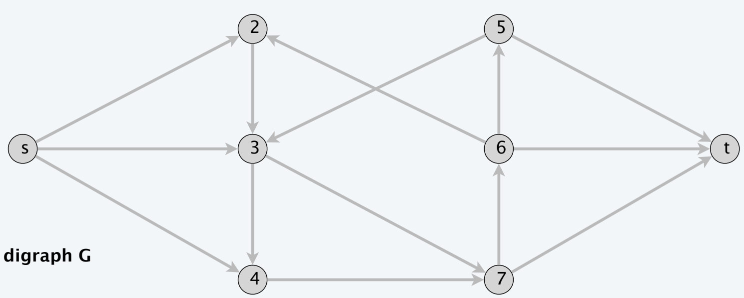
- bipartite matching
- **disjoint paths**
- extensions to max flow
- survey design
- airline scheduling
- image segmentation
- project selection
- baseball elimination

Edge-disjoint paths

Def. Two paths are edge-disjoint if they have no edge in common.

Edge-disjoint paths problem. Given a digraph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint $s \rightsquigarrow t$ paths.

Ex. Communication networks.



digraph G

Edge-disjoint paths

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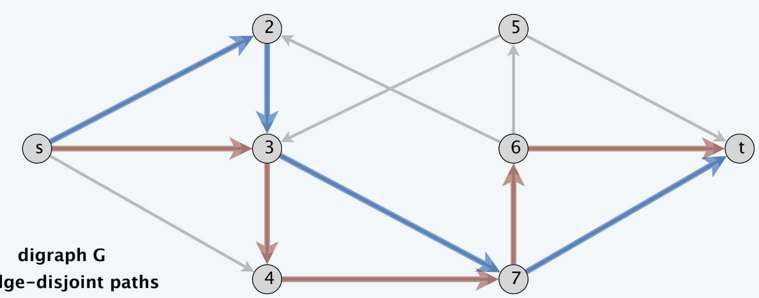
Edge-disjoint paths problem. Given a digraph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint $s \rightsquigarrow t$ paths.

Ex. Communication networks.



digraph G
2 edge-disjoint paths

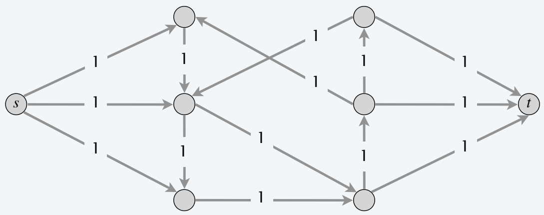
Edge-disjoint paths

Max-flow formulation. Assign unit capacity to every edge.

Theorem. 1–1 correspondence between k edge-disjoint $s \rightsquigarrow t$ paths in G and integral flows of value k in G'.

Pf. $\Rightarrow$

- Let $P_1, \ldots, P_k$ be k edge-disjoint $s \rightsquigarrow t$ paths in G.
- Set $f(e) = \begin{cases} 1 & \text{edge } e \text{ participates in some path } P_j \\ 0 & \text{otherwise} \end{cases}$
- Since paths are edge-disjoint, f is a flow of value k. ▪



Edge-disjoint paths

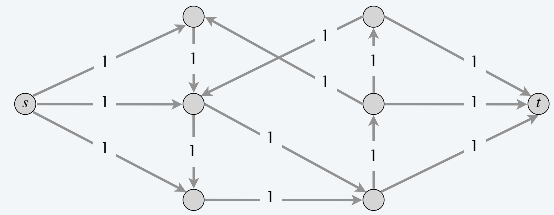
Max-flow formulation. Assign unit capacity to every edge.

Theorem. 1–1 correspondence between k edge-disjoint $s \rightsquigarrow t$ paths in G and integral flows of value k in G'.

Pf. $\Leftarrow$

- Let f be an integral flow in G' of value k.
- Consider edge (s, u) with $f(s, u) = 1$.
  - by flow conservation, there exists an edge (u, v) with $f(u, v) = 1$
  - continue until reach t, always choosing a new edge
- Produces k (not necessarily simple) edge-disjoint paths. ▪

can eliminate cycles to get simple paths in $O(mn)$ time if desired (flow decomposition)



Edge-disjoint paths

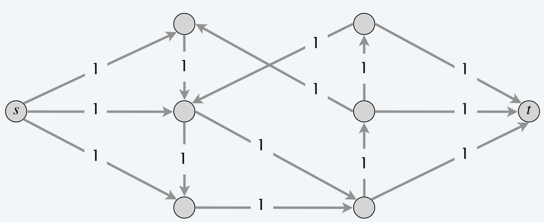
Max-flow formulation. Assign unit capacity to every edge.

Theorem. 1–1 correspondence between k edge-disjoint $s \rightsquigarrow t$ paths in G and integral flows of value k in G'.

Corollary. Can solve edge-disjoint paths problem via max-flow formulation.

Pf.

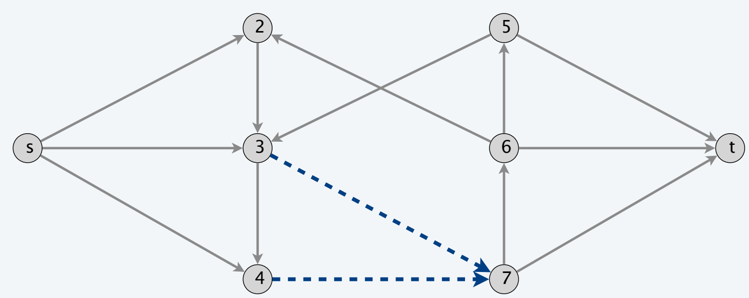
- Integrality theorem $\Rightarrow$ there exists a max flow f^* in G' that is integral.
- 1–1 correspondence $\Rightarrow$ f^* corresponds to max number of edge-disjoint $s \rightsquigarrow t$ paths in G. ▪



Network connectivity

Def. A set of edges $F \subseteq E$ disconnects t from s if every $s \rightsquigarrow t$ path uses at least one edge in F.

Network connectivity. Given a digraph $G = (V, E)$ and two nodes s and t, find min number of edges whose removal disconnects t from s.

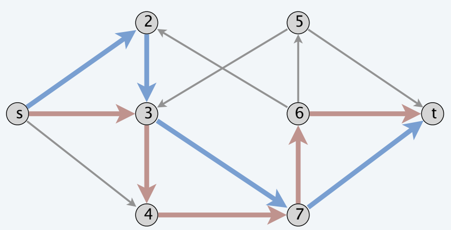


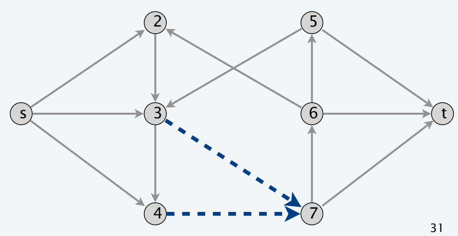
Menger's theorem

Theorem. [Menger 1927] The max number of edge-disjoint $s \rightsquigarrow t$ paths equals the min number of edges whose removal disconnects t from s.

Pf. $\leq$

- Suppose the removal of $F \subseteq E$ disconnects t from s, and $|F| = k$.
- Every $s \rightsquigarrow t$ path uses at least one edge in F.
- Hence, the number of edge-disjoint paths is $\leq k$. ▪



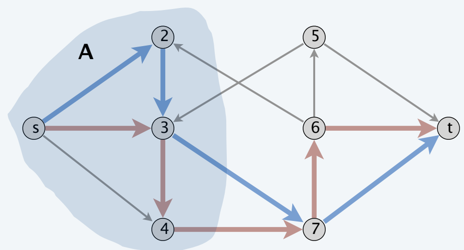


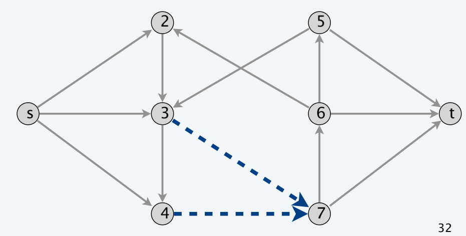
Menger's theorem

Theorem. [Menger 1927] The max number of edge-disjoint $s \rightsquigarrow t$ paths equals the min number of edges whose removal disconnects t from s.

Pf. $\geq$

- Suppose max number of edge-disjoint $s \rightsquigarrow t$ paths is k.
- Then value of max flow = k.
- Max-flow min-cut theorem $\Rightarrow$ there exists a cut (A, B) of capacity k.
- Let F be set of edges going from A to B.
- $|F| = k$ and disconnects t from s. ▪





Network flow II: quiz 3

How to find the max number of edge-disjoint paths in an undirected graph?

A. Solve the edge-disjoint paths problem in a digraph
(by replacing each undirected edge with two antiparallel edges).

B. Solve a max flow problem in an undirected graph.

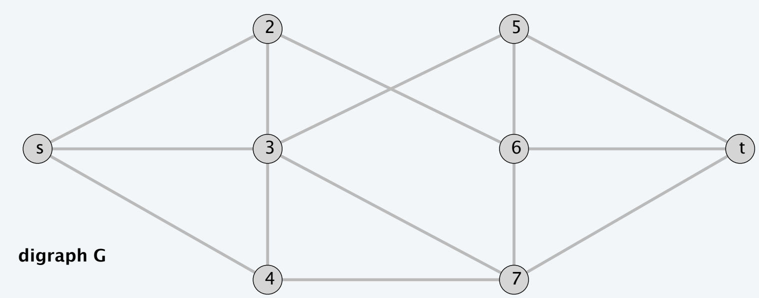
C. Both A and B.

D. Neither A nor B.

Edge-disjoint paths in undirected graphs

Def. Two paths are edge-disjoint if they have no edge in common.

Edge-disjoint paths problem in undirected graphs. Given a graph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint s–t paths.



Edge-disjoint paths in undirected graphs

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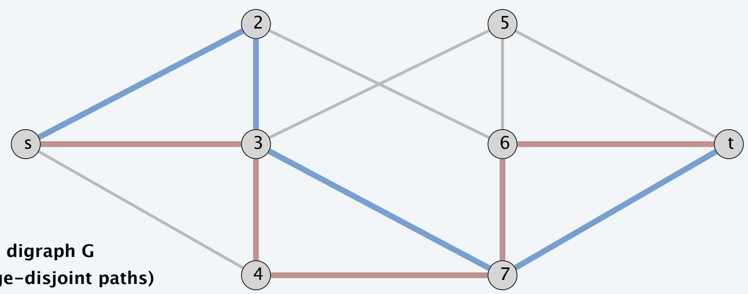
Edge-disjoint paths problem in undirected graphs. Given a graph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint s–t paths.

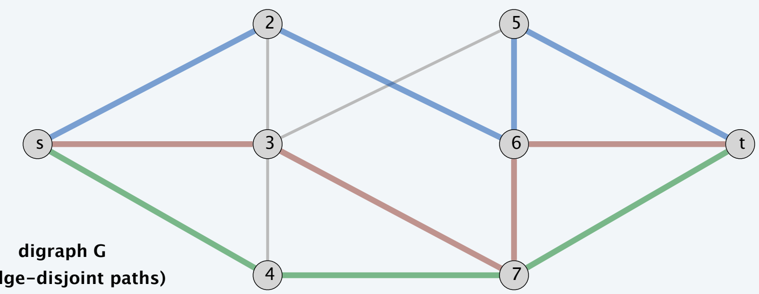


digraph G
(2 edge-disjoint paths)

Edge-disjoint paths in undirected graphs

Def. Two paths are edge-disjoint if they have no edge in common.

Edge-disjoint paths problem in undirected graphs. Given a graph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint s–t paths.



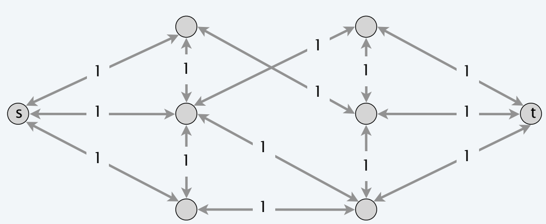
digraph G
(3 edge-disjoint paths)

Edge-disjoint paths in undirected graphs

Max-flow formulation. Replace each edge with two antiparallel edges and assign unit capacity to every edge.

Observation. Two paths P_1 and P_2 may be edge-disjoint in the digraph but not edge-disjoint in the undirected graph.

if P_1 uses edge (u, v)
and P_2 uses its antiparallel edge (v, u)



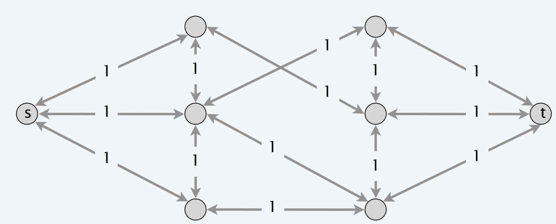
Edge-disjoint paths in undirected graphs

Max-flow formulation. Replace each edge with two antiparallel edges and assign unit capacity to every edge.

Lemma. In any flow network, there exists a maximum flow f in which for each pair of antiparallel edges e and e' : either $f(e) = 0$ or $f(e') = 0$ or both. Moreover, integrality theorem still holds.

Pf. [by induction on number of such pairs]

- Suppose $f(e) > 0$ and $f(e') > 0$ for a pair of antiparallel edges e and e'.
- Set $f(e) = f(e) - \delta$ and $f(e') = f(e') - \delta$, where $\delta = \min \{ f(e), f(e') \}$.
- f is still a flow of the same value but has one fewer such pair. ▪

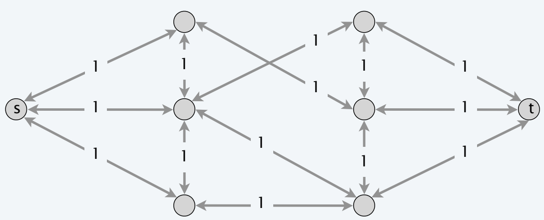


Edge-disjoint paths in undirected graphs

Max-flow formulation. Replace each edge with two antiparallel edges and assign unit capacity to every edge.

Lemma. In any flow network, there exists a maximum flow f in which for each pair of antiparallel edges e and e' : either $f(e) = 0$ or $f(e') = 0$ or both. Moreover, integrality theorem still holds.

Theorem. Max number of edge-disjoint $s \rightsquigarrow t$ paths = value of max flow.
Pf. Similar to proof in digraphs; use lemma.

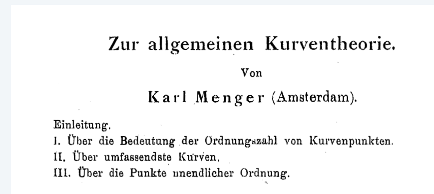


More Menger theorems

Theorem. Given an undirected graph and two nodes s and t, the max number of edge-disjoint s–t paths equals the min number of edges whose removal disconnects s and t.

Theorem. Given an undirected graph and two nonadjacent nodes s and t, the max number of internally node-disjoint s–t paths equals the min number of internal nodes whose removal disconnects s and t.

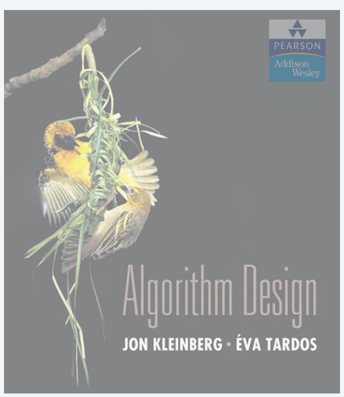
Theorem. Given a directed graph with two nonadjacent nodes s and t, the max number of internally node-disjoint $s \rightsquigarrow t$ paths equals the min number of internal nodes whose removal disconnects t from s.


Zur allgemeinen Kurventheorie.

Von

Karl Menger (Amsterdam).

Einleitung.
I. Über die Bedeutung der Ordnungszahl von Kurvenpunkten.
II. Über umfassendste Kurven.
III. Über die Punkte unendlicher Ordnung.



Section 7.7

7. Network Flow II

Network flow II: quiz 4

Which extensions to max flow can be easily modeled?

A. Multiple sources and multiple sinks.

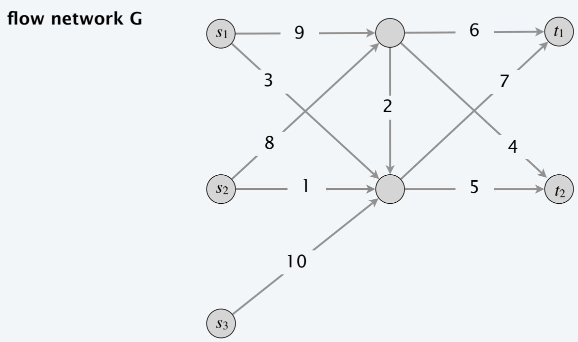
B. Undirected graphs.

C. Lower bounds on edge flows.

D. All of the above.

Multiple sources and sinks

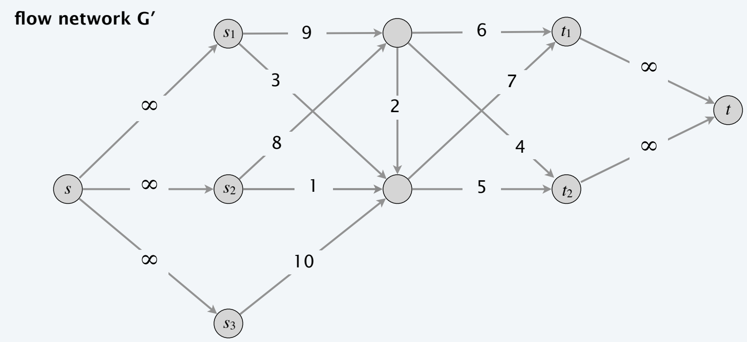
Def. Given a digraph $G = (V, E)$ with edge capacities $c(e) \geq 0$ and multiple source nodes and multiple sink nodes, find max flow that can be sent from the source nodes to the sink nodes.



Multiple sources and sinks: max-flow formulation

- Add a new source node s and sink node t.
- For each original source node s_i add edge (s, s_i) with capacity ∞.
- For each original sink node t_j, add edge (t_j, t) with capacity ∞.

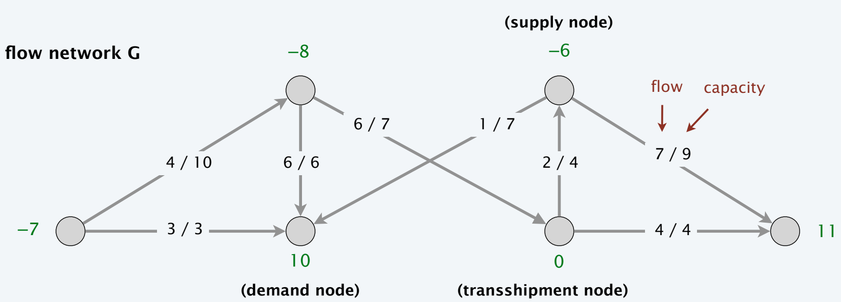
Claim. 1–1 correspondence between flows in G and G'.



Circulation with supplies and demands

Def. Given a digraph $G = (V, E)$ with edge capacities $c(e) \geq 0$ and node demands $d(v)$, a circulation is a function $f(e)$ that satisfies:

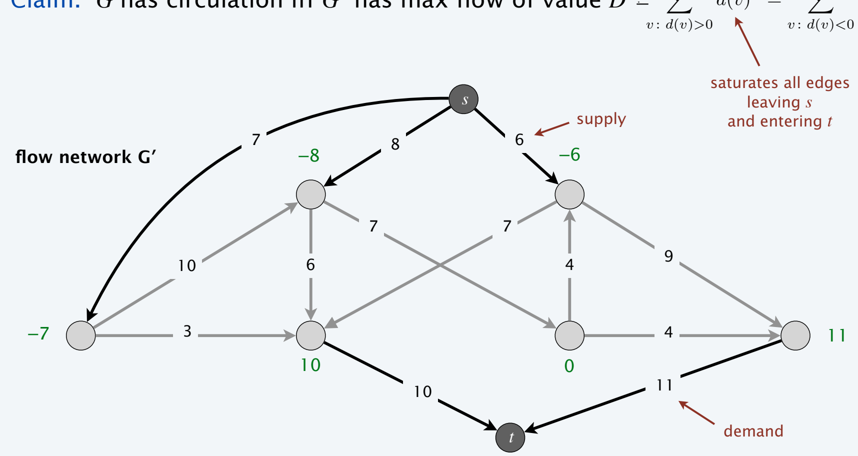
- For each $e \in E$: $\quad 0 \leq f(e) \leq c(e)$ (capacity)
- For each $v \in V$: $\quad \sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (flow conservation)



Circulation with supplies and demands: max-flow formulation

- Add new source s and sink t.
- For each v with $d(v) < 0$, add edge (s, v) with capacity $-d(v)$.
- For each v with $d(v) > 0$, add edge (v, t) with capacity $d(v)$.

Claim. G has circulation iff G' has max flow of value $D = \sum_{v \,:\, d(v) > 0} d(v) = \sum_{v \,:\, d(v) < 0} -d(v)$



Circulation with supplies and demands

Integrality theorem. If all capacities and demands are integers, and there exists a circulation, then there exists one that is integer-valued.

Pf. Follows from max-flow formulation + integrality theorem for max flow.

Theorem. Given (V, E, c, d), there does not exist a circulation iff there exists a node partition (A, B) such that $\Sigma_{v \in B} d(v) > cap(A, B)$.

demand by nodes in B exceeds supply of nodes in B plus max capacity of edges going from A to B

Pf sketch. Look at min cut in G'.

Circulation with supplies, demands, and lower bounds

Def. Given a digraph $G = (V, E)$ with edge capacities $c(e) \geq 0$, lower bounds $\ell(e) \geq 0$, and node demands $d(v)$, a circulation $f(e)$ is a function that satisfies:

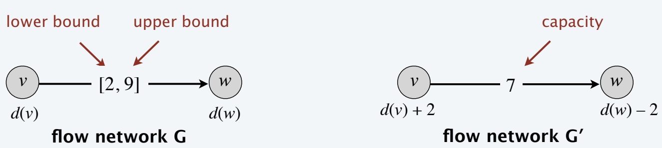
- For each $e \in E$: $\boxed{\ell(e) \leq f(e)} \leq c(e)$ (capacity)
- For each $v \in V$: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (flow conservation)

Circulation problem with lower bounds. Given (V, E, ℓ, c, d), does there exist a feasible circulation?

Circulation with supplies, demands, and lower bounds

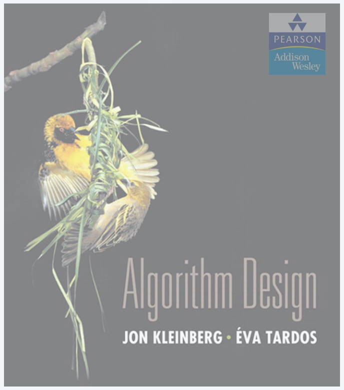
Max-flow formulation. Model lower bounds as circulation with demands.

- Send $\ell(e)$ units of flow along edge e.
- Update demands of both endpoints.



Theorem. There exists a circulation in G iff there exists a circulation in G'. Moreover, if all demands, capacities, and lower bounds in G are integers, then there exists a circulation in G that is integer-valued.

Pf sketch. $f(e)$ is a circulation in G iff $f'(e) = f(e) - \ell(e)$ is a circulation in G'.



SECTION 7.8

7. NETWORK FLOW II

- bipartite matching
- disjoint paths
- extensions to max flow
- survey design
- airline scheduling
- image segmentation
- project selection
- baseball elimination

Survey design

- Design survey asking n_1 consumers about n_2 products. ← one survey question per product
- Can survey consumer i about product j only if they own it.
- Ask consumer i between c_i and c_i' questions.
- Ask between p_j and p_j' consumers about product j.

Goal. Design a survey that meets these specs, if possible.

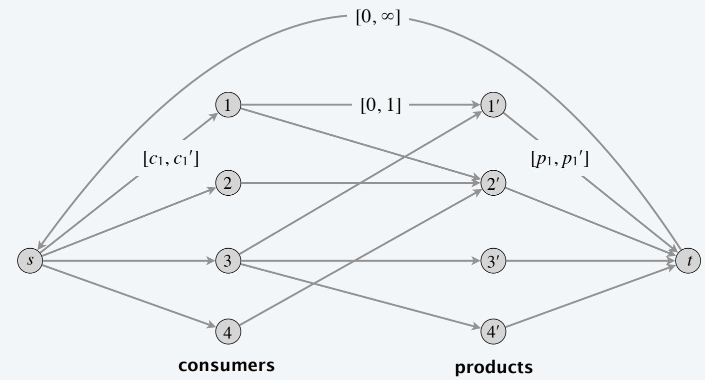
Bipartite perfect matching. Special case when $c_i = c_i' = p_j = p_j' = 1$.

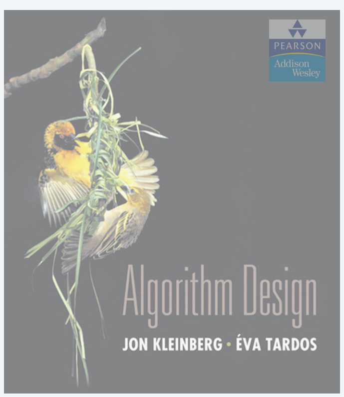
Survey design

Max-flow formulation. Model as a circulation problem with lower bounds.

all supplies and demands are 0

- Add edge (i, j) if consumer j owns product i.
- Add edge from s to consumer j.
- Add edge from product i to t.
- Add edge from t to s.
- All demands = 0.
- Integer circulation $\Longleftrightarrow$ feasible survey design.





SECTION 7.9

7. Network Flow II

- bipartite matching
- disjoint paths
- extensions to max flow
- survey design
- *airline scheduling*
- image segmentation
- project selection
- baseball elimination

Airline scheduling

Airline scheduling.

- Complex computational problem faced by airline carriers.
- Must produce schedules that are efficient in terms of equipment usage, crew allocation, and customer satisfaction. ← even in presence of unpredictable events, such as weather and breakdowns
- One of largest consumers of high-powered algorithmic techniques.

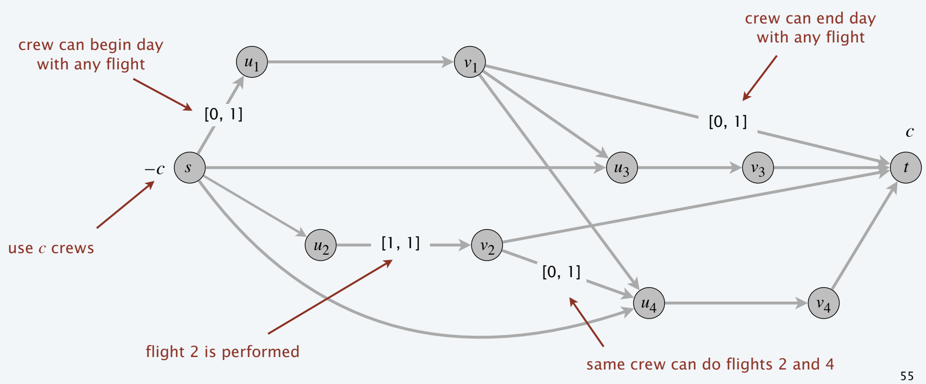
"Toy problem."

- Manage flight crews by reusing them over multiple flights.
- Input: set of k flights for a given day.
- Flight i leaves origin o_i at time s_i and arrives at destination d_i at time f_i.
- Minimize number of flight crews.

Airline scheduling

Circulation formulation. [to see if c crews suffice]

- For each flight i, include two nodes u_i and v_i. ← u_i = start of flight i, v_i = end of flight i
- Add source s with demand $-c$, and edges (s, u_i) with capacity 1.
- Add sink t with demand c, and edges (v_i, t) with capacity 1.
- For each i, add edge (u_i, v_i) with lower bound and capacity 1.
- if flight j reachable from i, add edge (v_i, u_j) with capacity 1.



Airline scheduling: running time

Theorem. The airline scheduling problem can be solved in $O(k^3 \log k)$ time.

Pf.

- k = number of flights.
- c = number of crews (unknown).
- $O(k)$ nodes, $O(k^2)$ edges.
- At most k crews needed.
  $\Rightarrow$ solve $\log_2 k$ circulation problems. ← binary search for min value c^*
- Value of any flow is between 0 and k.
  $\Rightarrow$ at most k augmentations per circulation problem.
- Overall time = $O(k^3 \log k)$.

Remark. Can solve in $O(k^3)$ time by formulating as minimum-flow problem.

Airline scheduling: postmortem

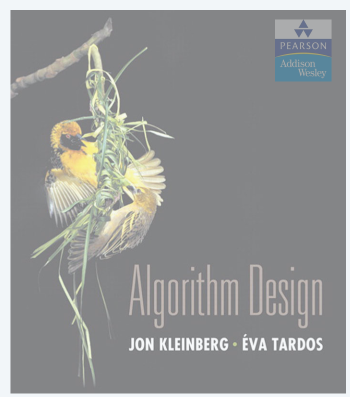
Remark. We solved a toy version of a real problem.

Real-world problem models countless other factors:

- Union regulations: e.g., flight crews can fly only a certain number of hours in a given time window.
- Need optimal schedule over planning horizon, not just one day.
- Deadheading has a cost.
- Flights don't always leave or arrive on schedule.
- Simultaneously optimize both flight schedule and fare structure.

Message.

- Our solution is a generally useful technique for efficient reuse of limited resources but trivializes real airline scheduling problem.
- Flow techniques useful for solving airline scheduling problems (and are widely used in practice).
- Running an airline efficiently is a very difficult problem.



Section 7.10

7. Network Flow II

- bipartite matching
- disjoint paths
- extensions to max flow
- survey design
- airline scheduling
- *image segmentation*
- project selection
- baseball elimination

Image segmentation

Image segmentation.

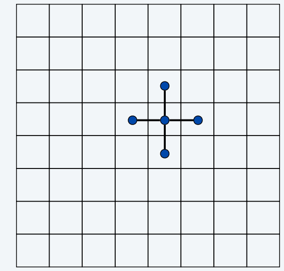
- Divide image into coherent regions.
- Central problem in image processing.

Ex. Separate human and robot from background scene.

Image segmentation

Foreground / background segmentation.

- Label each pixel in picture as belonging to foreground or background.
- V = set of pixels, E = pairs of neighboring pixels.
- $a_i \geq 0$ is likelihood pixel i in foreground.
- $b_i \geq 0$ is likelihood pixel i in background.
- $p_{ij} \geq 0$ is separation penalty for labeling one of i and j as foreground, and the other as background.



Goals.

- Accuracy: if $a_i > b_i$ in isolation, prefer to label i in foreground.
- Smoothness: if many neighbors of i are labeled foreground, we should be inclined to label i as foreground.
- Find partition (A, B) that maximizes: $$\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$

foreground background

Image segmentation

Formulate as min-cut problem.

- Maximization.
- No source or sink.
- Undirected graph.

Turn into minimization problem.

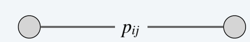
- Maximizing $$\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$
- is equivalent to minimizing $$\underbrace{\left(\sum_{i \in V} a_i + \sum_{j \in V} b_j \right)}_{\text{a constant}} - \sum_{i \in A} a_i - \sum_{j \in B} b_j + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$
- or alternatively $$\sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$

Image segmentation

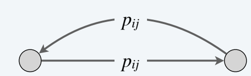
Formulate as min-cut problem $G' = (V', E')$.

- Include node for each pixel.
- Use two antiparallel edges instead of undirected edge.
- Add source s to correspond to foreground.
- Add sink t to correspond to background.

edge in G



two antiparallel edges in G′



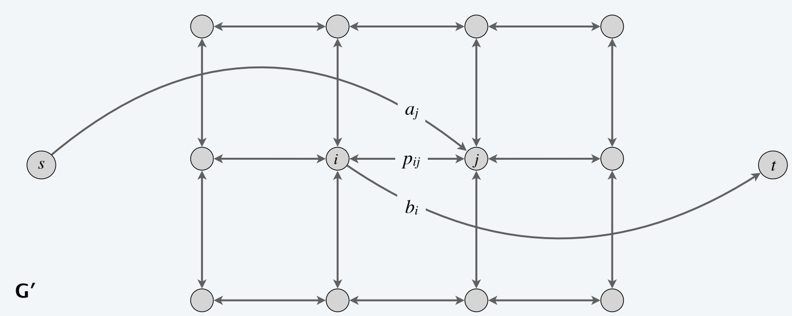


Image segmentation

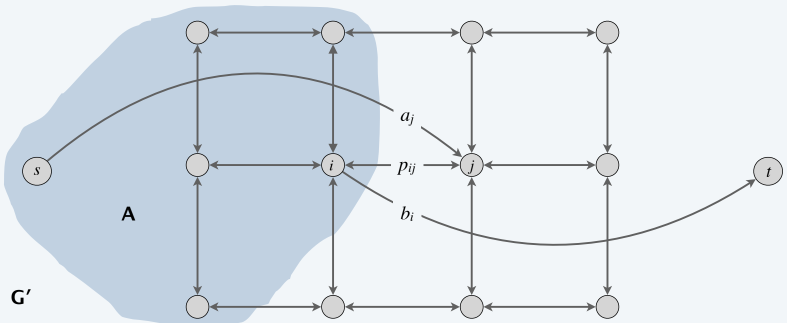
Consider min cut (A, B) in G'.

- A = foreground.

$$cap(A,B) \;=\; \sum_{j \in B} a_j \;+\; \sum_{i \in A} b_i \;+\; \sum_{\substack{(i,j) \in E \\ i \in A,\, j \in B}} p_{ij}$$

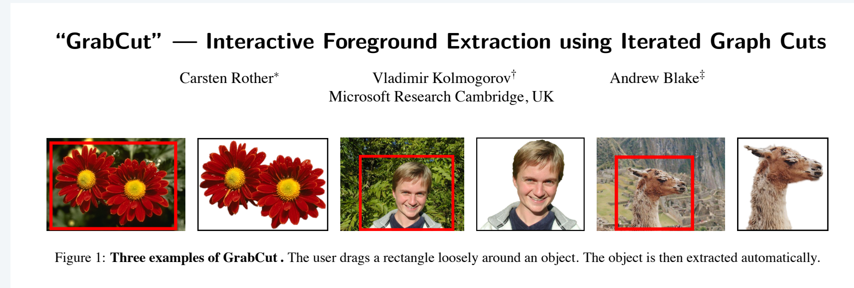
if i and j on different sides, p_{ij} counted exactly once

- Precisely the quantity we want to minimize.



Grabcut image segmentation

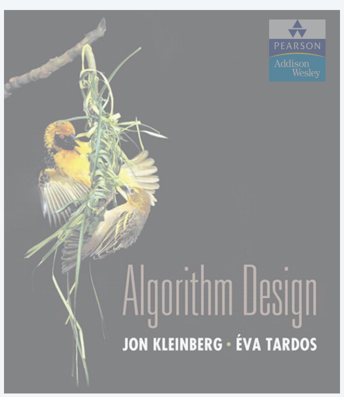
Grabcut. [Rother–Kolmogorov–Blake 2004]



"GrabCut" — Interactive Foreground Extraction using Iterated Graph Cuts

Carsten Rother* Vladimir Kolmogorov† Andrew Blake‡
Microsoft Research Cambridge, UK

Figure 1: **Three examples of GrabCut .** The user drags a rectangle loosely around an object. The object is then extracted automatically.



Section 7.11

7. Network Flow II

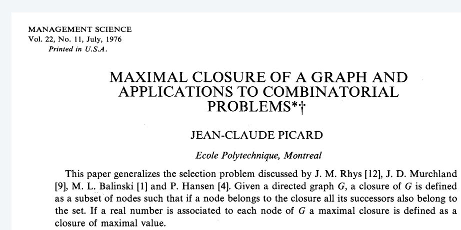
Project selection (maximum weight closure problem)

Projects with prerequisites.

can be positive or negative

- Set of possible projects P: project v has associated revenue p_v.
- Set of prerequisites E: $(v, w) \in E$ means w is a prerequisite for v.
- A subset of projects $A \subseteq P$ is feasible if the prerequisite of every project in A also belongs to A.

Project selection problem. Given a set of projects P and prerequisites E, choose a feasible subset of projects to maximize revenue.



MANAGEMENT SCIENCE
Vol. 22, No. 11, July, 1976
Printed in U.S.A.

MAXIMAL CLOSURE OF A GRAPH AND APPLICATIONS TO COMBINATORIAL PROBLEMS*†

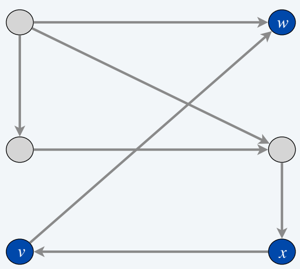
JEAN-CLAUDE PICARD

Ecole Polytechnique, Montreal

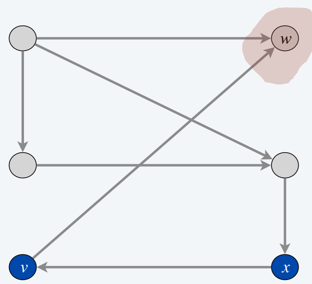
This paper generalizes the selection problem discussed by J. M. Rhys [12], J. D. Murchland [9], M. L. Balinski [1] and P. Hansen [4]. Given a directed graph G, a closure of G is defined as a subset of nodes such that if a node belongs to the closure all its successors also belong to the set. If a real number is associated to each node of G a maximal closure is defined as a closure of maximal value.

Project selection: prerequisite graph

Prerequisite graph. Add edge (v, w) if w is a prerequisite for v.



{ v, w, x } is feasible

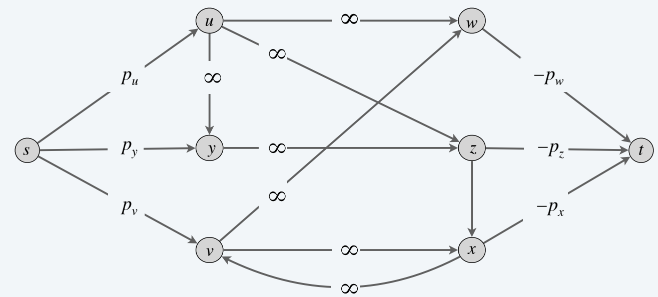


{ v, x } is infeasible

Project selection: min-cut formulation

Min-cut formulation.

- Assign a capacity of ∞ to each prerequisite edge.
- Add edge (s, v) with capacity p_v if $p_v > 0$.
- Add edge (v, t) with capacity $-p_v$ if $p_v < 0$.
- For notational convenience, define $p_s = p_t = 0$.



Project selection: min-cut formulation

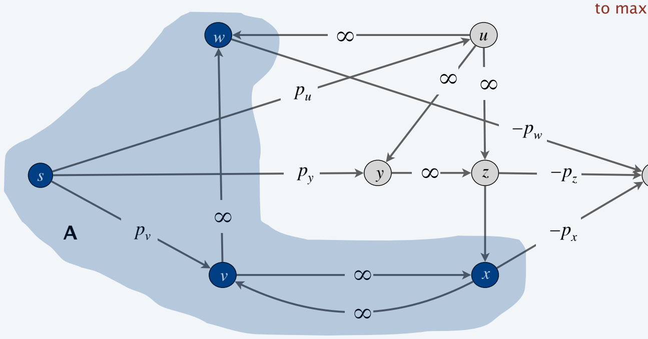
Claim. (A, B) is min cut iff $A - \{ s \}$ is an optimal set of projects.

- Infinite capacity edges ensure $A - \{ s \}$ is feasible.
- Max revenue because: $cap(A,B) \;=\; \sum_{v \in B:\; p_v > 0} p_v \;+\; \sum_{v \in A:\; p_v < 0} (-p_v)$

$$= \sum_{v:\; p_v > 0} p_v \;-\; \sum_{v \in A} p_v$$

a constant

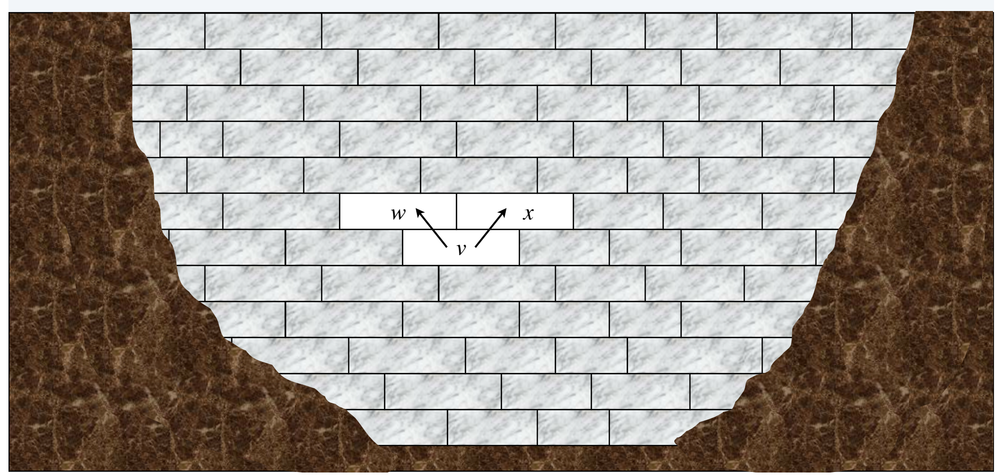
minimizing this is equivalent to maximizing revenue

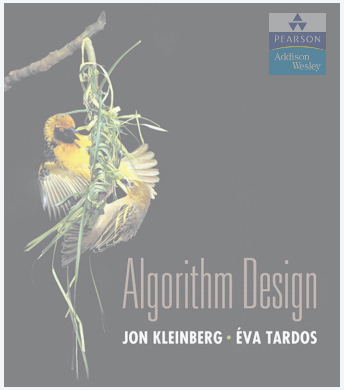


Open-pit mining

Open-pit mining. [studied since early 1960s]

- Blocks of earth are extracted from surface to retrieve ore.
- Each block v has net value p_v = value of ore – processing cost.
- Can't remove block v until both blocks w and x are removed.





SECTION 7.12

7. NETWORK FLOW II

- bipartite matching
- disjoint paths
- extensions to max flow
- survey design
- airline scheduling
- image segmentation
- project selection
- baseball elimination

Baseball elimination

TUESDAY, SEPTEMBER 10, 1996

San Francisco Chronicle

The Gate
Sports Online
http://www.sfgate.com

SPORTING G

49ers, Young Get Big Brea

Quarterback m

By Gary Swan
Chronicle Staff Writer

The bye week has come at a perfect time for the 49ers and quarterback Steve Young. If they had a game next Sunday, there's a good chance Young would not play.

But the pulled groin muscle on his up-

Giants Officially Leave the NL West Race

By Nancy Gay
Chronicle Staff Writer

With the smack of another National League West bat 500 miles away, the Giants' run at the division title ended last night, just as they were handing the visiting St. Louis Cardinals an even bigger lead in the NL Central.

CARDINALS 6
GIANTS 2

In San Diego, Greg Vaughn's three-run homer in the eighth pushed the Padres over the Pirates and officially shoved the rest of the Giants' season into the background. On the heels of their tedious 6-2 loss before an announced crowd of 10,307 at Candlestick Park, the Giants fell 19½ games off the lead.

As it is, the worst the Padres (80-65) can finish is 80-82. The Giants have fallen to 59-83 with 20 games left; they cannot win 80 games. Coming off a miserable 2-8 mark on a three-city road trip that saw their road record drop to 27-47, the Giants were hoping to get off on the right foot in their longest homestand of the year (15 games, 14 days).

Financing in Place
For Giants' New Stadium
SEE PAGE B1, MAIN NEWS

"Where we are, you're going to be eliminated sooner or later," Baker said quietly. "But it doesn't alter the fact that we've still got to play ball. You've still got to play hard, the fans come out to watch you play. You've got to play for the fact of loving to play, no matter where you are in the standings.

"You've got to play the role of spoiler, to not make it easier on

GIANTS: Page D5 Col. 3

Baseball elimination problem

Q. Which teams have a chance of finishing the season with the most wins?

i	team	wins	losses	to play	ATL	PHI	NYM	MON
0	Atlanta	83	71	8	–	1	6	1
1	Philly	80	79	3	1	–	0	2
2	New York	78	78	6	6	0	–	0
3	Montreal	77	82	3	1	2	0	–

Montreal is mathematically eliminated.

- Montreal finishes with ≤ 80 wins.
- Atlanta already has 83 wins.

Remark. This is the only reason sports writers appear to be aware of — conditions are sufficient but not necessary!

Baseball elimination problem

Q. Which teams have a chance of finishing the season with the most wins?

i	team	wins	losses	to play	ATL	PHI	NYM	MON
0	Atlanta	83	71	8	–	1	6	1
1	Philly	80	79	3	1	–	0	2
2	New York	78	78	6	6	0	–	0
3	Montreal	77	82	3	1	2	0	–

Philadelphia is mathematically eliminated.

- Philadelphia finishes with ≤ 83 wins.
- Either New York or Atlanta will finish with ≥ 84 wins.

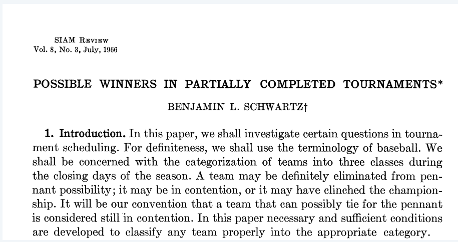
Observation. Answer depends not only on how many games already won and left to play, but on whom they're against.

Baseball elimination problem

Current standings.

- Set of teams S.
- Distinguished team $z \in S$.
- Team x has won w_x games already.
- Teams x and y play each other r_{xy} additional times.

Baseball elimination problem. Given the current standings, is there any outcome of the remaining games in which team z finishes with the most (or tied for the most) wins?



SIAM Review
Vol. 8, No. 3, July, 1966

POSSIBLE WINNERS IN PARTIALLY COMPLETED TOURNAMENTS*

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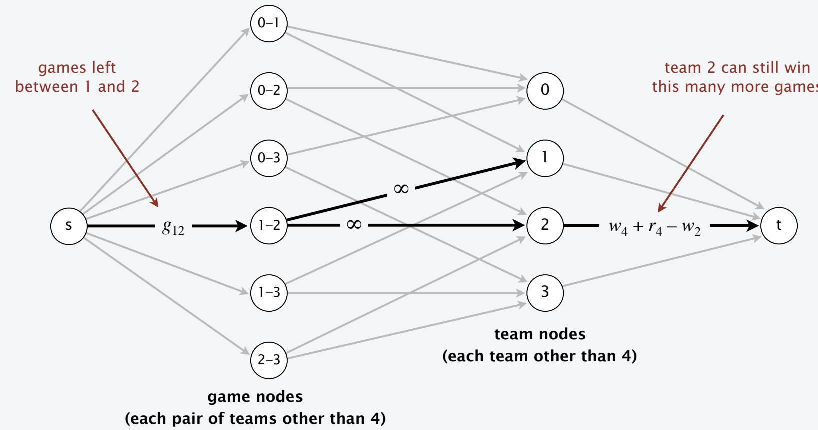
1. Introduction. In this paper, we shall investigate certain questions in tournament scheduling. For definiteness, we shall use the terminology of baseball. We shall be concerned with the categorization of teams into three classes during the closing days of the season. A team may be definitely eliminated from pennant possibility; it may be in contention, or it may have clinched the championship. It will be our convention that a team that can possibly tie for the pennant is considered still in contention. In this paper necessary and sufficient conditions are developed to classify any team properly into the appropriate category.

Baseball elimination problem: max-flow formulation

Can team 4 finish with most wins?

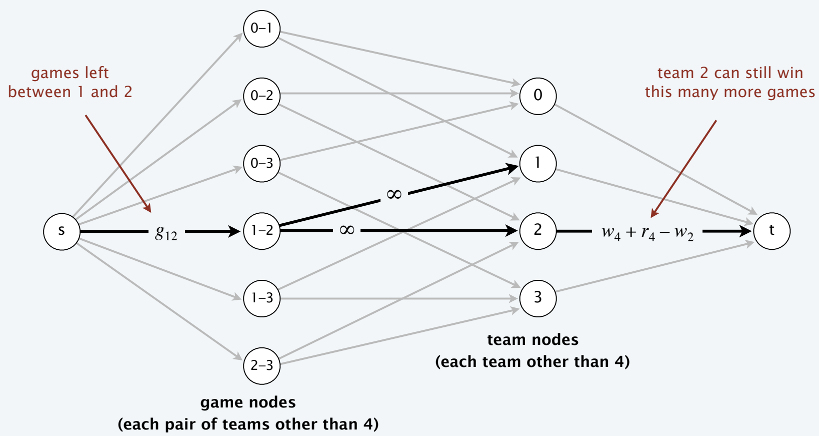
- Assume team 4 wins all remaining games $\Rightarrow$ $w_4 + r_4$ wins.
- Divvy remaining games so that all teams have $\leq w_4 + r_4$ wins.



Baseball elimination problem: max-flow formulation

Theorem. Team 4 not eliminated iff max flow saturates all edges leaving s.
Pf.

- Integrality theorem $\Rightarrow$ each remaining game between x and y added to number of wins for team x or team y.
- Capacity on (x, t) edges ensure no team wins too many games. ▪



Baseball elimination: explanation for sports writers

Q. Which teams have a chance of finishing the season with the most wins?

i		team	wins	losses	to play	NYY	BAL	BOS	TOR	DET
0		New York	75	59	28	–	3	8	7	3
1		Baltimore	71	63	28	3	–	2	7	4
2		Boston	69	66	27	8	2	–	0	0
3		Toronto	63	72	27	7	7	0	–	0
4		Detroit	49	86	27	3	4	0	0	–

AL East (August 30, 1996)

Detroit is mathematically eliminated.

- Detroit finishes with ≤ 76 wins.
- Wins for $R = \{ \text{NYY, BAL, BOS, TOR} \} = 278$.
- Remaining games among $\{ \text{NYY, BAL, BOS, TOR} \} = 3 + 8 + 7 + 2 + 7 = 27$.
- Average team in R wins $305/4 = 76.25$ games.

Baseball elimination: explanation for sports writers

Certificate of elimination.

$$T \subseteq S, \quad w(T) := \overbrace{\sum_{i \in T} w_i}^{\#\text{ wins}}, \quad g(T) := \overbrace{\sum_{\{x,y\} \subseteq T} g_{xy}}^{\#\text{ remaining games}},$$

Theorem. [Hoffman–Rivlin 1967] Team z is eliminated iff there exists a subset T^* such that

$$w_z + g_z \;<\; \frac{w(T^*) + g(T^*)}{|T^*|}$$

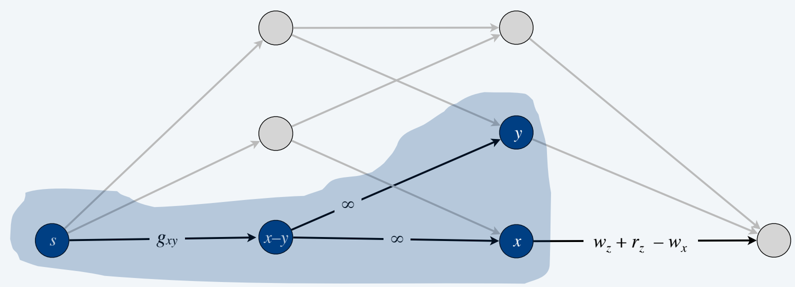
Pf. $\Leftarrow$

- Suppose there exists $T^* \subseteq S$ such that $w_z + g_z < \frac{w(T^*) + g(T^*)}{|T^*|}$.
- Then, the teams in T^* win at least $(w(T^*) + g(T^*)) / |T^*|$ games on average.
- This exceeds the maximum number that team z can win. ▪

Baseball elimination: explanation for sports writers

Pf. $\Rightarrow$

- Use max-flow formulation, and consider min cut (A, B).
- Let T^* = team nodes on source side A of min cut.
- Observe that game node $x\text{–}y \in A$ iff both $x \in T^*$ and $y \in T^*$.
  - infinite capacity edges ensure if $x\text{–}y \in A$, then both $x \in A$ and $y \in A$
  - if $x \in A$ and $y \in A$ but $x\text{–}y \notin A$, then adding $x\text{–}y$ to A decreases the capacity of the cut by g_{xy}



Baseball elimination: explanation for sports writers

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- Let T^* = team nodes on source side A of min cut.
- Observe that game node $x\text{–}y \in A$ iff both $x \in T^*$ and $y \in T^*$.
- Since team z is eliminated, by max-flow min-cut theorem,

$$\begin{aligned} g(S - \{z\}) \;&>\; cap(A, B) \\ &= \overbrace{g(S - \{z\}) - g(T^*)}^{\text{capacity of game edges leaving s}} \;+\; \overbrace{\sum_{x \in T^*} (w_z + g_z - w_x)}^{\text{capacity of team edges entering t}} \\ &= g(S - \{z\}) - g(T^*) \;-\; w(T^*) \;+\; |T^*|(w_z + g_z) \end{aligned}$$

- Rearranging terms: $w_z + g_z < \frac{w(T^*) + g(T^*)}{|T^*|}$ ▪