Well-posedness of the limiting equation of a noisy consensus model in opinion dynamics

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Abstract

This paper establishes the global well-posedness of the nonlinear Fokker–Planck equation for a noisy version of the Hegselmann–Krause model. The equation captures the mean-field behavior of a classic multi-agent system for opinion dynamics. We prove the global existence, uniqueness, nonnegativity and regularity of the weak solution. We also exhibit a global stability condition, which delineates a forbidden region for consensus formation. This is the first nonlinear stability result derived for the Hegselmann–Krause model.
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1. Introduction

Network-based dynamical systems feature agents that communicate via a dynamic graph while acting on the information they receive. These systems have received increasing attention lately because of their versatile use in modeling social and biological systems [1–7]. Typically, they consist of a fixed number N of agents, each one located at $x_k(t)$ on the real line. The agents'

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positions evolve as interactions take place along the edges of a dynamical graph that evolves endogenously. The motivation behind the model is to get a better understanding of the dynamics of collective behavior. Following [8,9], we express the system as a set of N coupled stochastic differential equations:

$$\mathrm{d}x_i = -\frac{1}{N}\sum_{j=1}^{N} a_{ij}(x_i - x_j)\mathrm{d}t + \sigma\,\mathrm{d}W_t^{(i)}, \tag{1}$$

where σ is the magnitude of the noise, $W_t^{(i)}$ are independent Wiener processes, and the "influence" parameter a_{ij} is a function of the distance between agents i and j; in other words, $a_{ij} = a(|x_i - x_j|)$, where a is nonnegative (to create attractive forces) and compactly supported over a fixed interval (to keep the range of the forces finite). Intuitively, the model mediates the competing tension between two opposing forces: the sum in (1) pulls the agents toward one another while the diffusion term keeps them jiggling in a Brownian motion; the two terms push the system into ordered and disordered states respectively. In the mean field limit, $N \to +\infty$, Equation (1) induces a nonlinear Fokker–Planck equation for the agent density profile $\rho(x, t)$ [8]:

$$\rho_t(x, t) = \left(\rho(x, t)\int \rho(x - y, t)ya(|y|)\mathrm{d}y\right)_x + \frac{\sigma^2}{2}\rho_{xx}(x, t). \tag{2}$$

The function $\rho(x, t)$ is the limit density of $\rho^N(x, t) := \frac{1}{N}\sum \delta_{x_j(t)}(dx)$, as N goes to infinity, where $\delta_x(dx)$ denotes the Dirac measure with point mass at x. The Fokker–Planck equation is a basic model in many areas of physics and it is a deterministic one describing how probability density functions evolve in time. A number of mathematical results, including well-posedness theory and the convergence to steady state as time $t \to \infty$, has been obtained for linear and nonlinear variants of the Fokker–Planck equation (see [10–17] and references therein).

This paper is concerned with the nonlinear, nonlocal Fokker–Planck equation (2) governing the density evolution of the noisy Hegselmann–Krause (*HK*) model. In the classic Hegselmann–Krause model, one of the most popular systems in consensus dynamics [18–20], each one of the N agents moves, at each time step, to the mass center of all the others within a fixed distance. The position of an agent represents its "opinion". If we add noise to this process, we obtain the discrete-time version of (1) for $a(y) = \mathbf{1}_{[0,R]}(y)$, where $\mathbf{1}_A$ is the usual set indicator function, equal to 1 if $y \in A$ and 0 otherwise. To be exact, the original *HK* model does not scale a_{ij} by $1/N$ but by the reciprocal of the number agents within distance R of agent i. Canuto et al. [21] have argued that this difference has a minor impact on the dynamics. By preserving the pairwise symmetry among the agents, however, the formulation (1) simplifies the analysis.

The *HK* model has been the subject of extensive investigation. A sample of the literature includes work on convergence and consensus properties [1,22,18,23–27], conjectures about the spatial features of the attractor set [28], and various extensions such as *HK* systems with inertia [29], leaders [30–32], or random jumps [33].

Some mathematical models for opinion formation have been proposed under different assumptions. A class of kinetic models of opinion formation and its asymptotic limit Fokker–Planck type equations are derived in [34], based on two-body interactions involving compromise and diffusion properties in exchanges between individuals. Later, based on [34], a mathematical model for opinion formation in a society that is built of two groups, one groups of "ordinary" people and one group of "strong opinion leaders", and its corresponding Fokker–Planck type

equations are proposed in [35]. Some other opinion dynamics models are obtained in [36]. However, fewer rigorous mathematical results including the global well-posedness and stability of the models are studied so far. In [37], the blow-up of the smooth solution of an aggregation equations with mildly singular interaction kernels is proved. This model is a continuum one of the kinetic equations in the modeling of animal collective behavior such as flocks, schools or swarms of insects, fishes and birds, which shares some feature with the Patlak–Keller–Segel model for chemotaxis (see [38]).

The present paper focuses on the analysis of the nonlinear Fokker–Planck equation for the noisy *HK* system, which corresponds to setting $a(y) = \mathbf{1}_{[0,R]}(y)$ in (2). For concreteness, let us denote $U = [-\ell, \ell]$, $U_T = [-\ell, \ell] \times (0, T)$ for $T > 0$, and consider the following periodic problem for the *HK* system:

$$\begin{cases} \rho_t - \dfrac{\sigma^2}{2}\rho_{xx} = \left(\rho G_\rho\right)_x & \text{in } U_T \\ \rho(\ell, \cdot) = \rho(-\ell, \cdot) & \text{on } \partial U \times [0, T] \\ \rho = \rho_0 & \text{on } U \times \{t = 0\} \end{cases} \tag{3}$$

where

$$G_\rho(x, t) := \int\limits_{x-R}^{x+R} (x - y)\,\rho(y, t)\,dy \tag{4}$$

and the initial condition ρ_0 is assumed to be a probability density, i.e., $\rho_0 \geq 0$ and $\int_U \rho_0(x)\,dx = 1$. The positive constants ℓ, R and σ are fixed with $0 < R < \ell$. Note that we have to periodically extend ρ outside of U in order to make sense of the integral above. The periodicity of ρ, together with Eq. (3), immediately implies the normalization condition

$$\int\limits_U \rho(y, t)\,dy = 1 \tag{5}$$

for all $0 \leq t < T$.

Main results. We establish the global well-posedness of Eq. (3), which entails the existence, uniqueness, nonnegativity and regularity of the solution. In addition, we prove a global stability condition for the uniform solution $\rho = \frac{1}{2\ell}$, representing the state without any clustering of opinions. This gives a sufficient condition involving R and σ for which no consensus can be reached regardless of the initial condition.

The paper is organized as follows. In Section 2, we first derive the aforementioned global stability condition by assuming that a sufficiently smooth solution exists. More precisely, we show that as long as $\sigma^2 > \frac{2\ell}{\pi}\left(2R + R^2/\sqrt{3}\,\ell\right)$, the uniform solution $\rho = \frac{1}{2\ell}$ is unconditionally stable in the sense that $\rho(t) \to \frac{1}{2\ell}$ exponentially as $t \to \infty$ for any initial data $\rho_0 \in L^2_{per}(U)$. Instead of employing the usual entropy method (see e.g. [11]), we use energy estimates to directly obtain L^2_{per} exponential convergence. An important ingredient in the proof is a L^1_{per} estimate on the solution (Lemma 2.1). Interestingly, this estimate immediately implies the nonnegativity of the solution while no arguments using maximum principles are required. The latter may not be easy to obtain for nonlinear partial integro-differential equations, such as the one we consider

here. We close the section by discussing the physical significance of the stability result and how it relates to other works in the opinion dynamics literature.

In Section 3, we prove the global existence and uniqueness of the weak solution to (3) when the initial data $\rho_0 \in L^2_{per}(U)$. Here, we construct approximate solutions by considering a sequence of linear parabolic equations obtained from (3) by replacing ρG_ρ with $\rho_n G_{\rho_{n-1}}$. Using energy estimates, we find that the sequence of solutions ρ_n forms a Cauchy sequence in $L^1(0, T; L^1_{per}(U))$ and we use this strong convergence result to simplify the existence proof.

Finally, in Section 4 we establish improved regularity properties of the weak solution if $\rho_0 \in H^k_{per}(U)$ for some $k \geq 1$. This allows us to remove the a priori smoothness assumptions in the stability and positivity results of Section 2. The techniques employed in this section are standard in the analysis of partial differential equations. The main results in this paper are summarized in Theorem 4.5.

Notation. As customary in the literature, we often treat ρ (and other functions on U_T) not as a function from U_T to $\mathbb{R}$, but from $[0, T]$ to a relevant Banach space. In this case, we define for each t,

$$[\rho(t)](\cdot) := \rho(\cdot, t). \tag{6}$$

For a shorthand, we denote the usual L^p norms on U by

$$\|\rho(t)\|_p := \|\rho(t)\|_{L^p(U)} \tag{7}$$

Other Banach space norms will be written out explicitly. Since we are dealing with periodic boundary conditions, a subscript "per" will often be attached to the relevant Banach space symbols to denote the subspace of periodic functions on U, e.g.,

$$
\begin{aligned}
L^p_{per}(U) &:= \left\{ u \in L^p(U) : u(-\ell) = u(+\ell) \right\}, \\
H^k_{per}(U) &:= \left\{ u \in H^k(U) : u(-\ell) = u(+\ell) \right\},
\end{aligned}
\tag{8}
$$

for $1 \leq p \leq \infty$ and $k \geq 1$. They are equipped with the usual norms.

Finally, we denote by $C, C_1, C_2, \ldots$ any generic constants (possibly depending on R, ℓ or σ) used in the various energy estimates. Their actual values may change from line to line. When they depend on the time interval under consideration, we will indicate the dependence explicitly, e.g., $C(T)$.

2. Nonnegativity and global stability via a priori estimates

First, let us assume that there exists a sufficiently smooth solution

$$\rho \in C^1\left(0, \infty; C^2_{per}(U)\right), \tag{9}$$

to equation (3). This allows us to prove a priori energy estimates, from which we can deduce the solution's nonnegativity and derive a global stability condition of the spatially uniform solution $\rho = \frac{1}{2\ell}$.

We begin by setting $\psi = \rho - \frac{1}{2\ell}$ so that $\int_U \psi(y, t)\, dy = 0$. Then, ψ satisfies the equation

$$\psi_t - \frac{\sigma^2}{2}\psi_{xx} = \left(\psi G_\psi\right)_x + \frac{1}{2\ell}\left(G_\psi\right)_x.$$ (10)

The other two extra terms are zero since $\int_{x-R}^{x+R}(x-y)\,dy = 0$ for all x. Multiplying equation (10) by ψ and integrating by parts over U, we have

$$\frac{1}{2}\frac{d}{dt}\|\psi(t)\|_2^2 + \frac{\sigma^2}{2}\|\psi_x(t)\|_2^2 \le \left|\int_U \psi_x\psi G_\psi dx\right| + \frac{1}{2\ell}\left|\int_U \psi_x G_\psi dx\right|.$$ (11)

By the Cauchy–Schwarz inequality,

$$\frac{1}{2}\frac{d}{dt}\|\psi(t)\|_2^2 + \frac{\sigma^2}{2}\|\psi_x(t)\|_2^2 \le \left\|G_\psi(t)\right\|_\infty\|\psi(t)\|_2\|\psi_x(t)\|_2$$

$$+\frac{1}{2\ell}\|\psi_x(t)\|_2\left\|G_\psi(t)\right\|_2.$$ (12)

First let us estimate $\left\|G_\psi(t)\right\|_\infty$. For any x and t, we have

$$\left|G_\psi(x,t)\right| \le \int_U |y-x|\mathbf{1}_{\{|y-x|\le R\}}|\psi(y,t)|\,dy$$

$$\le R\|\psi(t)\|_1$$

$$\le R\left(1+\|\rho(t)\|_1\right).$$ (13)

In order to proceed with the bound on $\left\|G_\psi(t)\right\|_\infty$, we need an L^1_{per} estimate of $\rho(t)$.

Lemma 2.1. *Suppose* $\rho \in C^1\left(0,\infty; C^2_{per}(U)\right)$ *is a solution of (3) with* $\rho_0 \in L^1_{per}(U)$, *then* $\|\rho(t)\|_1 \le \|\rho_0\|_1$ *for all* $t \ge 0$.

Proof. Let $\epsilon > 0$ and define

$$\chi_\epsilon(r) = \begin{cases} |r| & |r| > \epsilon \\ -\frac{r^4}{8\epsilon^3} + \frac{3r^2}{4\epsilon} + \frac{3\epsilon}{8} & |r| \le \epsilon \end{cases}.$$ (14)

This is a convex C^2-approximation of the absolute value function satisfying $|r| \le \chi_\epsilon(r)$. Multiplying $\chi'_\epsilon(\rho)$ to equation (3) and integrating by parts over U, we have

$$\frac{d}{dt}\int_U \chi_\epsilon(\rho(x,t))\,dx + \frac{\sigma^2}{2}\left\|\rho_x(t)\left[\chi''_\epsilon(\rho(t))\right]^{1/2}\right\|_2^2 = -\int_U \rho G_\rho \chi''_\epsilon(\rho)\rho_x\,dx.$$ (15)

By Cauchy–Schwarz and Young's inequality,

$$\frac{d}{dt} \int_U \chi_\epsilon \left(\rho \left(x, t \right) \right) dx + \frac{\sigma^2}{2} \left\| \rho_x \left(t \right) \left[\chi_\epsilon'' \left(\rho \left(t \right) \right) \right]^{1/2} \right\|_2^2$$

$$\leq \left\| \rho \left(t \right) G_\rho \left(t \right) \left[\chi_\epsilon'' \left(\rho \left(t \right) \right) \right]^{1/2} \right\|_2 \left\| \rho_x \left(t \right) \left[\chi_\epsilon'' \left(\rho \left(t \right) \right) \right]^{1/2} \right\|_2$$

$$\leq \frac{1}{2\sigma^2} \left\| \rho \left(t \right) G_\rho \left(t \right) \left[\chi_\epsilon'' \left(\rho \left(t \right) \right) \right]^{1/2} \right\|_2^2 + \frac{\sigma^2}{2} \left\| \rho_x \left(t \right) \left[\chi_\epsilon'' \left(\rho \left(t \right) \right) \right]^{1/2} \right\|_2^2. \tag{16}$$

Therefore,

$$\frac{d}{dt} \int_U \chi_\epsilon \left(\rho \left(x, t \right) \right) dx \leq \frac{1}{2\sigma^2} \left\| \rho \left(t \right) G_\rho \left(t \right) \left[\chi_\epsilon'' \left(\rho \left(t \right) \right) \right]^{1/2} \right\|_2^2$$

$$\leq \frac{1}{2\sigma^2} \left\| G_\rho \left(t \right) \right\|_\infty^2 \left\| \rho \left(t \right) \left[\chi_\epsilon'' \left(\rho \left(t \right) \right) \right]^{1/2} \right\|_2^2. \tag{17}$$

Replacing ψ by ρ in (13) we have

$$\left\| G_\rho \left(t \right) \right\|_\infty \leq R \left\| \rho \left(t \right) \right\|_1. \tag{18}$$

The term $\left\| \rho \left(t \right) \left[\chi_\epsilon'' \left(\rho \left(t \right) \right) \right]^{1/2} \right\|_2^2$ can be split into two integrals:

$$\left\| \rho \left(t \right) \left[\chi_\epsilon'' \left(\rho \left(t \right) \right) \right]^{1/2} \right\|_2^2 = \int_U \rho^2 \chi_\epsilon'' \left(\rho \right) dx$$

$$= \int_U \rho^2 \chi_\epsilon'' \left(\rho \right) \mathbf{1}_{\{|\rho|>\epsilon\}} dx$$

$$+ \int_U \rho^2 \chi_\epsilon'' \left(\rho \right) \mathbf{1}_{\{|\rho|\leq\epsilon\}} dx. \tag{19}$$

For $|\rho| > \epsilon$, $\chi_\epsilon'' \left(\rho \right) = 0$ by construction, and hence the first integral above is zero. The second integral is estimated as:

$$\int_U \rho^2 \chi_\epsilon'' \left(\rho \right) \mathbf{1}_{\{|\rho|\leq\epsilon\}} dx = \int_U \frac{3\rho^2 \left(\epsilon^2 - \rho^2 \right)}{2\epsilon^3} \mathbf{1}_{\{|\rho|\leq\epsilon\}} dx$$

$$\leq \int_U \frac{3\epsilon}{2} dx = 3\ell\epsilon. \tag{20}$$

Therefore, by (18), (19), (20), Eq. (17) becomes

$$\frac{d}{dt} \int_U \chi_\epsilon \left(\rho \left(x, t \right) \right) dx \leq \frac{3\ell R^2 \epsilon}{2\sigma^2} \left\| \rho \left(t \right) \right\|_1^2 \leq \frac{3\ell R^2 \epsilon}{2\sigma^2} \left[\int_U \chi_\epsilon \left(\rho \left(x, t \right) \right) dx \right]^2.$$

Applying Grönwall's inequality, we get

$$\int_U \chi_\epsilon \left(\rho \left(x, t \right) \right) dx \leq \left(\int_U \chi_\epsilon \left(\rho_0 \left(x \right) \right) dx \right)$$

$$\times \exp \left[\frac{3 \ell R^2 \epsilon}{2 \sigma^2} \int_0^t \int_U \chi_\epsilon \left(\rho \left(x, s \right) \right) dx \, ds \right]. \tag{21}$$

Since ρ is continuous, the integral in the exponential is finite. Therefore, taking the limit $\epsilon \to 0$ yields

$$\| \rho \left(t \right) \|_1 \leq \| \rho_0 \|_1, \tag{22}$$

for every $t \geq 0$, as required. $\square$

Incidentally, Lemma 2.1 establishes the nonnegativity of ρ. This is important because ρ represents the density of opinions of individuals and, as such, is necessarily nonnegative at all times. It is interesting that a L^1_{per} estimate suffices to show nonnegativity and no arguments from maximum principles are required.

Corollary 2.2. *If $\rho \in C^1 \left(0, \infty; C^2_{per} \left(U \right) \right)$ is a solution of (3), with $\rho_0 \geq 0$ and $\int_U \rho_0 \left(x \right) dx = 1$, then $\| \rho \left(t \right) \|_1 = 1$ and $\rho \left(t \right) \geq 0$ in U for all $t \geq 0$.*

Proof. Since $\int_U \rho_0 \left(x \right) dx = 1$, the normalization condition (5) is satisfied for $t > 0$. Applying Lemma 2.1, we have

$$1 = \int_U \rho \left(x, t \right) dx \leq \| \rho \left(t \right) \|_1 \leq \| \rho_0 \|_1 = \int_U \rho_0 \, dx = 1. \tag{23}$$

Hence, $\| \rho \left(t \right) \|_1 = 1$. But

$$1 = \int_U \rho \left(x, t \right) dx = \int_U \rho \mathbf{1}_{\{\rho \geq 0\}} dx + \int_U \rho \mathbf{1}_{\{\rho < 0\}} dx, \tag{24}$$

$$1 = \int_U | \rho \left(x, t \right) | dx = \int_U \rho \mathbf{1}_{\{\rho \geq 0\}} dx - \int_U \rho \mathbf{1}_{\{\rho < 0\}} dx. \tag{25}$$

These equations imply that $\int_U \rho \mathbf{1}_{\{\rho < 0\}} dx = 0$, and hence, $\rho \left(t \right) \geq 0$ a.e. in U. By continuity, $\rho \left(t \right) \geq 0$ in U for all $t \geq 0$. $\square$

With Lemma 2.1, it follows from (13) that

$$\left\| G_\psi \left(t \right) \right\|_\infty \leq 2R \tag{26}$$

Next, we also have

$$
\left(G_\psi\left(x, t\right)\right)^2 = \left(\int_{x-R}^{x+R}(x-y)\,\psi\left(y, t\right) dy\right)^2
$$

$$
\leq \int_{x-R}^{x+R}(x-y)^2\,dy \int_{x-R}^{x+R}(\psi\left(y, t\right))^2\,dy
$$

$$
= \frac{2}{3}R^3 \int_{x-R}^{x+R}(\psi\left(y, t\right))^2\,dy. \tag{27}
$$

Consequently,

$$
\left\|G_\psi\left(t\right)\right\|_2^2 \leq \frac{2}{3}R^3 \int_U \int_{x-R}^{x+R}(\psi\left(y, t\right))^2\,dy dx
$$

$$
= \frac{2}{3}R^3 \int_U \int_{-R}^{+R}(\psi\left(x+y, t\right))^2\,dy dx
$$

$$
= \frac{4R^4}{3}\left\|\psi\left(t\right)\right\|_2^2. \tag{28}
$$

With estimates (26) and (28), (12) becomes

$$
\frac{1}{2}\frac{d}{dt}\left\|\psi\left(t\right)\right\|_2^2 + \frac{\sigma^2}{2}\left\|\psi_x\left(t\right)\right\|_2^2 \leq \left(2R + \frac{R^2}{\sqrt{3}\,\ell}\right)\left\|\psi\left(t\right)\right\|_2 \left\|\psi_x\left(t\right)\right\|_2
$$

$$
\leq \frac{\sigma^2}{4}\left\|\psi_x\left(t\right)\right\|_2^2
$$

$$
+ \frac{1}{\sigma^2}\left(2R + \frac{R^2}{\sqrt{3}\,\ell}\right)^2\left\|\psi\left(t\right)\right\|_2^2. \tag{29}
$$

Hence, we have

$$
\frac{1}{2}\frac{d}{dt}\left\|\psi\left(t\right)\right\|_2^2 \leq \frac{1}{\sigma^2}\left(2R + \frac{R^2}{\sqrt{3}\,\ell}\right)^2\left\|\psi\left(t\right)\right\|_2^2 - \frac{\sigma^2}{4}\left\|\psi_x\left(t\right)\right\|_2^2. \tag{30}
$$

By construction, $\int_U \psi\left(x, t\right) dx = 0$ for all t. Thus, $\psi\left(t\right)$ satisfies the Poincaré inequality

$$
\left\|\psi\left(t\right)\right\|_2 \leq C\left\|\psi_x\left(t\right)\right\|_2. \tag{31}
$$

For the interval $U = [-\ell, \ell]$, the optimal Poincaré constant is $C = \ell/\pi$. Therefore, (30) becomes

$$\frac{d}{dt}\|\psi(t)\|_2^2 \le \left(\frac{2}{\sigma^2}\left(2R+\frac{R^2}{\sqrt{3}\,\ell}\right)^2 - \frac{\pi^2\sigma^2}{2\ell^2}\right)\|\psi(t)\|_2^2. \tag{32}$$

But

$$\|\psi(0)\|_2^2 = \left\|\rho_0 - \frac{1}{2\ell}\right\|_2^2 \le 2\|\rho_0\|_2^2 + \frac{1}{\ell}. \tag{33}$$

Thus we obtain the integral form of (30):

$$\|\psi(t)\|_2^2 \le \left(2\|\rho_0\|_2^2 + \frac{1}{\ell}\right)\exp\left\{\left(\frac{2}{\sigma^2}\left(2R+\frac{R^2}{\sqrt{3}\,\ell}\right)^2 - \frac{\pi^2\sigma^2}{2\ell^2}\right)t\right\}. \tag{34}$$

In particular, if $\sigma^2 > \frac{2\ell}{\pi}\left(2R+R^2/\sqrt{3}\,\ell\right)$, the constant factor in the exponential is negative, therefore $\|\psi(t)\|_2^2 \to 0$ as long as $\|\rho_0\|_2 < \infty$. We summarize these results:

Theorem 2.3. *Let $\rho \in C^1\left(0,\infty; C^2_{per}(U)\right)$ be a solution of (3) with $\rho_0 \ge 0$, $\int_U \rho_0(x)\,dx = 1$, and $\rho_0 \in L^2_{per}(U)$. If $\sigma^2 > \frac{2\ell}{\pi}\left(2R+R^2/\sqrt{3}\,\ell\right)$, then $\rho(t) \to \frac{1}{2\ell}$ in L^2_{per} exponentially as $t \to \infty$.*

2.1. Physical significance of Theorem 2.3

The noisy *HK* model generally exhibits two types of steady-states. The first is a spatially uniform steady-state, i.e., ρ is constant. This represents the case where individuals have uniformly distributed opinions, without any local or global consensus. The second involves one or more clusters of individuals having similar opinions, in which case ρ is a multi-modal profile. Which of these two steady-states appear in the long-time limit depends on the interaction radius R and noise σ, as well as the initial profile ρ_0.

In this aspect, Theorem 2.3 gives a sufficient condition for the spatially uniform solution to be the globally attractive state, irrespective of the initial profile ρ_0. In other words, as long as $\sigma^2 > \frac{2\ell}{\pi}\left(2R+R^2/\sqrt{3}\,\ell\right)$, any initial profile converges to the spatially uniform state. In particular, clustered profiles do not even have local stability. This immediately indicates a forbidden zone for consensus: when the volatility of one's opinion is too large compared to the interaction radius, there can be no clustering of opinions regardless of the initial opinion distribution. It should be noted that this is the first result regarding the nonlinear stability of the *HK* system. On the other hand, it is possible to perform linear stability analysis of Eq. (3) at the uniform solution $\rho = \frac{1}{2\ell}$ to derive a linear stability condition for the uniform solution [39]. The main idea is to plug ansatz $\rho(x,t) = \frac{1}{2\ell} + p(t)e^{\frac{\pi x i}{\ell}}$ into Eq. (3) and ignore higher order terms to get a linear differential equation of $p(t)$, then locate the region of parameters (σ, R) where the trivial solution $p(t) = 0$ loses its stability. The combination of these two results indicate a region where it is possible to have both clustered and uniform states as locally stable solutions (see Fig. 1).

HK system with finite and infinite numbers of agents in both discrete- and continuous-time fashions exhibit similar behavior of opinion aggregation and bifurcation [40]. In particular, the competition between clustered and uniform solution is also observed in the simulations of
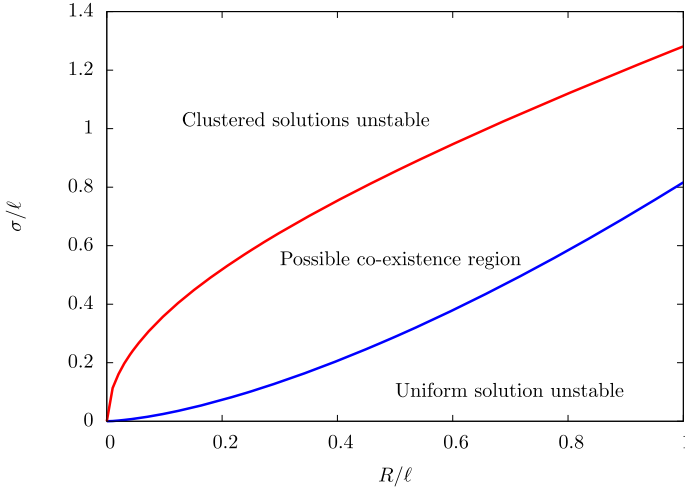
Fig. 1. The phase diagram for the noisy *HK* model. The top (red) curve is the curve $\sigma^2 = \frac{2\ell}{\pi}\left(2R + R^2/\sqrt{3}\,\ell\right)$, above which the spatially uniform solution ($\rho = 1/2\ell$) is unconditionally stable, i.e. no clustering of opinions is possible. The bottom (blue) curve is obtained from linear stability analysis around the spatially uniform solution, and has the form $(\sigma/\ell)^2 = C(R/\ell)^3$ [39]. Below this curve, the uniform solution loses linear stability and only clustered solutions are permitted. Between these two curves is the region for which both clustered and uniform solutions can be stable with respect to small perturbations. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

discrete-time noisy *HK* system with finite number of agents [33]. The advantage of considering the continuous-time *HK* system in its mean-field limit is that the stability of non-trivial solutions can be carried out analytically, as indicated by Theorem 2.3.

3. Existence and uniqueness

Our discussion so far has assumed the existence of a solution to (3). In this section, we prove the existence and uniqueness of the weak solution by defining a sequence of linear parabolic equations, whose solutions converge strongly to a function ρ that solves a weak formulation of Eq. (3). To begin with, let $T > 0$ and consider a sequence of linear parabolic equations

$$
\begin{cases}
\rho_{n_t} - \dfrac{\sigma^2}{2}\rho_{n_{xx}} = \left(\rho_n G_{\rho_{n-1}}\right)_x & \text{in } U_T \\
\rho_n\left(\ell, \cdot\right) = \rho_n\left(-\ell, \cdot\right) & \text{on } \partial U \times [0, T] \\
\rho_n = \rho_0 & \text{on } U \times \{t = 0\}
\end{cases}
\tag{35}
$$

for $n \geq 1$, with $\rho_0(x, t) := \rho_0(x)$ for all $t > 0$. For convenience, we assume that the initial condition satisfies $\rho_0 \in C^\infty_{per}(U)$, $\rho_0 \geq 0$ and $\int_U \rho_0(x)\,dx = 1$. The smoothness condition will be relaxed later (see Theorem 3.12 at the end of this section).

Consider the case $n = 1$. Since $\rho_0 \in C^\infty_{per}(U)$ and both G_{ρ_0} and $\left(G_{\rho_0}\right)_x$ are bounded, by standard results on linear parabolic evolution equations, there exists a unique $\rho_1 \in C^\infty\left(0, T; C^\infty_{per}(U)\right)$ satisfying (35) for $n = 1$. Iterating this for $n > 1$ implies that there exists a sequence of smooth functions $\{\rho_n : n \geq 1\}$ satisfying (35). Next, we establish some uniform energy estimates on ρ_n.

Proposition 3.1. *Let $T > 0$ and suppose $\{\rho_n : n \geq 1\}$ satisfy (35) with $\rho_0 \in C^\infty_{per}(U)$. Then, $\|\rho_n(t)\|_1 \leq \|\rho_0\|_1$ for all $0 \leq t \leq T$ and $n \geq 1$.*

Proof. Since we know that $\rho_n(t) \in C^\infty\left(0, T; C^\infty_{per}(U)\right)$ for all $n \geq 1$ and all $0 \leq t \leq T$, we can proceed exactly as in the proof of Lemma 2.1. In this case, instead of (21) we have

$$\int_U \chi_\epsilon(\rho_n(x, t)) \, dx \leq \left(\int_U \chi_\epsilon(\rho_0(x)) \, dx \right)$$

$$\times \exp\left[\frac{3\ell R^2 \epsilon}{2\sigma^2} \int_0^t \int_U \chi_\epsilon(\rho_{n-1}(x, s)) \, dx \, ds \right]. \tag{36}$$

Since ρ_{n-1} is smooth, the integral in the exponential is finite, hence we take the limit $\epsilon \to 0$ to obtain

$$\|\rho_n(t)\|_1 \leq \|\rho_0\|_1, \tag{37}$$

for all $0 \leq t \leq T$. $\quad\square$

Corollary 3.2. *Let $T > 0$ and suppose $\{\rho_n : n \geq 1\}$ satisfy (35) with $\rho_0 \in C^\infty_{per}(U)$, $\rho_0 \geq 0$ and $\int_U \rho_0(x) \, dx = 1$. Then, $\|\rho_n(t)\|_1 = 1$ and $\rho_n(t) \geq 0$ in U for all $0 \leq t \leq T$ and for all $n \geq 1$.*

Proof. Since the functions ρ_n are all periodic, we have $\int_U \rho_n(x, t) \, dx = 1$; hence the proof is identical to Corollary 2.2. $\quad\square$

Proposition 3.3. *Let $T > 0$ and suppose $\{\rho_n : n \geq 1\}$ satisfy (35) with $\rho_0 \in C^\infty_{per}(U)$, $\rho_0 \geq 0$ and $\int_U \rho_0(x) \, dx = 1$. Then, there exists a constant $C(T) > 0$ such that*

$$\|\rho_n\|_{L^\infty\left(0, T; L^2_{per}(U)\right)} + \|\rho_n\|_{L^2\left(0, T; H^1_{per}(U)\right)} \leq C(T) \|\rho_0\|_2.$$

Proof. We proceed as in Section 2 by multiplying (35) by ρ_n and integrating by parts. This gives us

$$\frac{1}{2} \frac{d}{dt} \|\rho_n(t)\|_2^2 + \frac{\sigma^2}{2} \|\rho_{n_x}(t)\|_2^2 \leq \int_U \left| \rho_n \rho_{n_x} G_{\rho_{n-1}} \right| dx$$

$$\leq \left\| \rho_{n_x}(t) \right\|_2 \left\| \rho_n(t) G_{\rho_{n-1}}(t) \right\|_2$$

$$\leq \frac{\sigma^2}{4} \left\| \rho_{n_x}(t) \right\|_2^2$$

$$+ \frac{1}{\sigma^2} \|\rho_n(t)\|_2^2 \left\| G_{\rho_{n-1}}(t) \right\|_\infty^2. \tag{38}$$

Using Proposition 3.1 and Corollary 3.2 we have

$$\left\| G_{\rho_{n-1}}(t) \right\|_{\infty} \le R \left\| \rho_{n-1}(t) \right\|_1 = R, \tag{39}$$

and hence (38) becomes

$$\frac{1}{2} \frac{d}{dt} \left\| \rho_n(t) \right\|_2^2 + \frac{\sigma^2}{4} \left\| \rho_{n_x}(t) \right\|_2^2 \le \frac{R}{\sigma^2} \left\| \rho_n(t) \right\|_2^2, \tag{40}$$

which implies, by integration, that

$$\left\| \rho_n(t) \right\|_2^2 \le C(T) \left\| \rho_0 \right\|_2^2, \tag{41}$$

for all $0 \le t \le T$, and

$$\sup_{0 \le t \le T} \left\| \rho_n(t) \right\|_2^2 + \int_0^T \left\| \rho_{n_x}(t) \right\|_2^2 dt \le C(T) \left\| \rho_0 \right\|_2^2. \qquad \square \tag{42}$$

Proposition 3.4. *Let $T > 0$ and suppose $\{\rho_n : n \ge 1\}$ satisfy (35) with $\rho_0 \in C_{per}^{\infty}(U)$, $\rho_0 \ge 0$ and $\int_U \rho_0(x)\,dx = 1$. Then, there exists a constant $C(T) > 0$ such that*

$$\left\| \rho_n \right\|_{L^{\infty}\left(0,T; H_{per}^1(U)\right)} + \left\| \rho_n \right\|_{L^2\left(0,T; H_{per}^2(U)\right)} \le C(T) \left(\left\| \rho_0 \right\|_{H_{per}^1(U)}^2 + \left\| \rho_0 \right\|_2^4 \right)^{1/2}. \tag{43}$$

Proof. Multiplying equation (35) by $-\rho_{n_{xx}}$ and integrating by parts over U, it follows from Cauchy–Schwarz, Young's inequality, and (39) that

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left\| \rho_{n_x}(t) \right\|_2^2 + \frac{\sigma^2}{2} \left\| \rho_{n_{xx}}(t) \right\|_2^2 \\
&\le \int_U \left| \left(\rho_n G_{\rho_{n-1}} \right)_x \rho_{n_{xx}} \right| dx \\
&\le \frac{\sigma^2}{4} \left\| \rho_{n_{xx}}(t) \right\|_2^2 + C_1 \left\| G_{\rho_{n-1}} \rho_{n_x}(t) \right\|_2^2 + C_2 \left\| \left(G_{\rho_{n-1}} \right)_x (t)\, \rho_n(t) \right\|_2^2 \\
&\le \frac{\sigma^2}{4} \left\| \rho_{n_{xx}}(t) \right\|_2^2 + C_1 \left\| \rho_{n_x}(t) \right\|_2^2 + C_2 \left\| \left(G_{\rho_{n-1}} \right)_x (t)\, \rho_n(t) \right\|_2^2.
\end{aligned} \tag{44}$$

Now,

$$\begin{aligned}
\left(G_{\rho_{n-1}} \right)_x &= \frac{\partial}{\partial x} \left[\int_{x-R}^{x+R} (x - y)\, \rho_{n-1}(y, t)\, dy \right] \\
&= -R \left(\rho_{n-1}(x + R, t) + \rho_{n-1}(x - R, t) \right) \\
&\quad + \int_{x-R}^{x+R} \rho_{n-1}(y, t)\, dy.
\end{aligned} \tag{45}$$

By (41) and Morrey's inequality,

$$
\begin{aligned}
\left\|\left(G_{\rho_{n-1}}\right)_x (t)\, \rho_n (t)\right\|_2^2 &\leq \|\rho_n (t)\|_\infty^2 \left\|\left(G_{\rho_{n-1}}\right)_x (t)\right\|_2^2 \\
&\leq C \|\rho_n (t)\|_{H^1_{per}(U)}^2 \|\rho_{n-1} (t)\|_2^2 \\
&\leq C(T) \|\rho_0\|_2^2 \|\rho_n (t)\|_{H^1_{per}(U)}^2 .
\end{aligned}
\tag{46}
$$

It follows that (44) becomes

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\|\rho_{n_x}(t)\right\|_2^2 + \frac{\sigma^2}{4}\left\|\rho_{n_{xx}}(t)\right\|_2^2 &\leq C_1 \left\|\rho_{n_x}(t)\right\|_2^2 \\
&\quad + C_2(T)\|\rho_0\|_2^2 \|\rho_n(t)\|_{H^1_{per}(U)}^2 .
\end{aligned}
\tag{47}
$$

Integrating over t, we have

$$
\begin{aligned}
\frac{1}{2}\sup_{0\leq t\leq T}\left\|\rho_{n_x}(t)\right\|_2^2 &+ \frac{\sigma^2}{4}\left\|\rho_{n_{xx}}\right\|_{L^2\left(0,T;L^2_{per}(U)\right)}^2 \\
&\leq \frac{1}{2}\left\|\rho_{0_x}\right\|_2^2 + C(T)\left(\|\rho_n\|_{L^2\left(0,T;H^1_{per}(U)\right)}^2 + \|\rho_0\|_2^2 \|\rho_n\|_{L^2\left(0,T;H^1_{per}(U)\right)}^2\right).
\end{aligned}
\tag{48}
$$

Applying the estimates in Proposition 3.3, we find that

$$
\sup_{0\leq t\leq T}\left\|\rho_{n_x}(t)\right\|_2^2 + \left\|\rho_{n_{xx}}\right\|_{L^2\left(0,T;L^2_{per}(U)\right)}^2 \leq C(T)\left(\|\rho_0\|_{H^1_{per}(U)}^2 + \|\rho_0\|_2^4\right). \qquad \square
\tag{49}
$$

With the uniform estimates above, we can now show that ρ_n converges strongly to a limit.

Lemma 3.5. Let $T > 0$ and suppose that $\{\rho_n : n \geq 1\}$ satisfies (35) with $\rho_0 \in C^\infty_{per}(U)$, $\rho_0 \geq 0$ and $\int_U \rho_0(x)\,dx = 1$. Then there exists $\rho \in L^1\left(0,T;L^1_{per}(U)\right)$ such that $\rho_n \to \rho$ in $L^1\left(0,T;L^1_{per}(U)\right)$.

Proof. We set $\phi_n = \rho_n - \rho_{n-1}$ for $n \geq 1$. For $n \geq 2$, the evolution equation for ϕ_n reads

$$
\phi_{n_t} - \frac{\sigma^2}{2}\phi_{n_{xx}} = \left(\phi_n G_{\rho_{n-1}} + \rho_{n-1} G_{\phi_{n-1}}\right)_x .
\tag{50}
$$

Let $\epsilon > 0$. Multiplying the equation above by $\chi'_\epsilon(\phi_n)$ (see definition (14)) and integrating by parts yield

$$
\begin{aligned}
\frac{d}{dt}\int_U \chi_\epsilon(\phi_n)\,dx &+ \frac{\sigma^2}{2}\left\|\left[\chi''_\epsilon(\phi_n(t))\right]^{1/2}\phi_{n_x}(t)\right\|_2^2 \\
&\leq \int_U \left|\chi''_\epsilon(\phi_n)\,\phi_{n_x}\phi_n G_{\rho_{n-1}}\right|dx + \int_U \left|\chi'_\epsilon(\phi_n)\left(\rho_{n-1}G_{\phi_{n-1}}\right)_x\right|dx
\end{aligned}
$$

$$\leq \frac{\sigma^2}{2} \left\| \left[\chi_\epsilon'' \left(\phi_n \left(t \right) \right) \right]^{1/2} \phi_{n_x} \left(t \right) \right\|_2^2$$

$$+ \frac{1}{2\sigma^2} \left\| G_{\rho_{n-1}} \left(t \right) \right\|_\infty^2 \left\| \left[\chi_\epsilon'' \left(\phi_n \left(t \right) \right) \right]^{1/2} \phi_n \left(t \right) \right\|_2^2$$

$$+ \int_U \left| \chi_\epsilon' \left(\phi_n \right) \left(\rho_{n-1} G_{\phi_{n-1}} \right)_x \right| dx. \tag{51}$$

By [Corollary 3.2](#), $\left\| G_{\rho_{n-1}} \left(t \right) \right\|_\infty \leq R \left\| \rho_{n-1} \left(t \right) \right\|_1 = R$. Also, as in [(20)](#) from the proof of Lemma 2.1, we have

$$\left\| \left[\chi_\epsilon'' \left(\phi_n \left(t \right) \right) \right]^{1/2} \phi_n \left(t \right) \right\|_2^2 \leq C\epsilon. \tag{52}$$

To estimate the last integral in [(51)](#), observe that $\left| \chi_\epsilon' \right| \leq C$, hence

$$\int_U \left| \chi_\epsilon' \left(\phi_n \right) \left(\rho_{n-1} G_{\phi_{n-1}} \right)_x \right| dx \leq C \int_U \left| \left(\rho_{n-1} \right)_x G_{\phi_{n-1}} \right| dx$$

$$+ C \int_U \left| \rho_{n-1} \left(G_{\phi_{n-1}} \right)_x \right| dx$$

$$\leq C_1 \left\| G_{\phi_{n-1}} \right\|_\infty \left\| \rho_{n-1} \right\|_{H^1_{per}(U)}$$

$$+ C_2 \left\| \rho_{n-1} \right\|_\infty \left\| \left(G_{\phi_{n-1}} \right)_x \right\|_1. \tag{53}$$

But we know that $\left\| G_{\phi_{n-1}} \right\|_\infty \leq R \left\| \phi_{n-1} \right\|_1$ and that $\left\| \left(G_{\phi_{n-1}} \right)_x \right\|_1 \leq C \left\| \phi_{n-1} \right\|_1$ (see expression [(45)](#)). Moreover, Morrey's inequality implies $\left\| \rho_{n-1} \right\|_\infty \leq \left\| \rho_{n-1} \right\|_{H^1_{per}(U)}$. Hence, it follows that, as ϵ tends to 0, [(51)](#) becomes

$$\frac{d}{dt} \left\| \phi_n \left(t \right) \right\|_1 \leq C \left\| \rho_{n-1} \left(t \right) \right\|_{H^1_{per}(U)} \left\| \phi_{n-1} \left(t \right) \right\|_1$$

$$\leq C \left(\rho_0; 1, T \right) \left\| \phi_{n-1} \left(t \right) \right\|_1, \tag{54}$$

where in the last line we used [Proposition 3.4](#) and the shorthand

$$C \left(\rho_0; 1, T \right) := C \left(T \right) \left(\left\| \rho_0 \right\|_{H^1_{per}(U)}^2 + \left\| \rho_0 \right\|_2^4 \right)^{1/2}. \tag{55}$$

Now, for $N \geq 2$ we define

$$y_N \left(t \right) := \sum_{n=2}^{N} \left\| \phi_n \left(t \right) \right\|_1. \tag{56}$$

By (54) and Corollary 3.2,

$$\frac{d}{dt} y_N(t) \leq C(\rho_0; 1, T) \left(y_N(t) + \|\phi_1(t)\|_1 - \|\phi_N(t)\|_1 \right)$$
$$\leq C(\rho_0; 1, T) (y_N(t) + 4). \tag{57}$$

Moreover, the ρ_n's coincide at $t = 0$, so $y_N(0) = 0$. Thus by Grönwall's inequality,

$$y_N(t) \leq 2TC(\rho_0; 1, T) e^{TC(\rho_0; 1, T)}, \tag{58}$$

uniformly in N and t. Furthermore, for each t, $y_N(t)$ is a bounded monotone sequence in N, hence there exists

$$y_\infty(t) = \sum_{n=2}^{\infty} \|\phi_n(t)\|_1 \leq 2TC(\rho_0; 1, T) e^{TC(\rho_0; 1, T)}, \tag{59}$$

such that $y_N(t) \uparrow y_\infty(t)$, pointwise in t. By the monotone convergence theorem,

$$\int_0^T y_N(t) \, dt \uparrow \int_0^T y_\infty(t) \, dt \leq 2T^2 C(\rho_0; 1, T) e^{TC(\rho_0; 1, T)}. \tag{60}$$

This result immediately implies that $\{\rho_n\}$ is a Cauchy sequence in $L^1\left(0, T; L^1_{per}(U)\right)$. Indeed, for $\epsilon > 0$ we can pick $N \geq 2$ such that $\int_0^T y_\infty(t) \, dt - \int_0^T y_N(t) \, dt < \epsilon$. Hence, for all $M \geq 1$,

$$\|\rho_{N+M} - \rho_N\|_{L^1\left(0, T; L^1_{per}(U)\right)} = \int_0^T \|\rho_{N+M}(t) - \rho_N(t)\|_1 \, dt$$

$$= \int_0^T \left\| \sum_{n=N+1}^{N+M} \phi_n(t) \right\|_1 dt$$

$$\leq \int_0^T \sum_{n=N+1}^{N+M} \|\phi_n(t)\|_1 \, dt$$

$$= \int_0^T y_{N+M}(t) \, dt - \int_0^T y_N(t) \, dt$$

$$\leq \int_0^T y_\infty(t) \, dt - \int_0^T y_N(t) \, dt$$

$$\leq \epsilon. \tag{61}$$

Therefore, $\{\rho_n\}$ is a Cauchy sequence and there exists $\rho \in L^1\left(0, T; L^1_{per}(U)\right)$ such that $\rho_n \to \rho$ in $L^1\left(0, T; L^1_{per}(U)\right)$. $\square$

Note that we can extract from $\{\rho_n\}$ a subsequence that converges weakly in smaller spaces.

Definition 3.6. We denote by $H^{-1}_{per}(U)$ the dual space of $H^1_{per}(U)$.

Since periodic boundary conditions allow integration by parts without extra terms, most characterizations of $H^{-1} = \left(H^1_0\right)^*$ carry over to H^{-1}_{per}.

Lemma 3.7. *We have $\rho \in L^2\left(0, T; H^1_{per}(U)\right) \cap L^\infty\left(0, T; L^2_{per}(U)\right)$, with $\rho_t \in L^2\left(0, T; H^{-1}_{per}(U)\right)$, and the estimate*

$$\|\rho\|_{L^\infty\left(0,T;L^2_{per}(U)\right)} + \|\rho\|_{L^2\left(0,T;H^1_{per}(U)\right)} + \|\rho_t\|_{L^2\left(0,T;H^{-1}_{per}(U)\right)} \leq C(T)\|\rho_0\|_2. \tag{62}$$

Moreover, there exists a subsequence $\left\{\rho_{n_k} : k \geq 1\right\}$ such that

$$\rho_{n_k} \rightharpoonup \rho \ \text{in } L^2\left(0, T; H^1_{per}\right),$$

and

$$\rho_{n_{k_t}} \rightharpoonup \rho_t \ \text{in } L^2\left(0, T; H^{-1}_{per}\right).$$

Proof. From Proposition 3.3, we have

$$\|\rho_n\|_{L^\infty\left(0,T;L^2_{per}(U)\right)} + \|\rho_n\|_{L^2\left(0,T;H^1_{per}(U)\right)} \leq C(T)\|\rho_0\|_2. \tag{63}$$

Next, observe that from the evolution equation of ρ_n, we have

$$\rho_{n_t} = \left(G_{\rho_{n-1}}\rho_n + \frac{\sigma^2}{2}\rho_{n_x}\right)_x. \tag{64}$$

Hence,

$$\|\rho_{n_t}\|^2_{L^2\left(0,T;H^{-1}_{per}(U)\right)} \leq \int_0^T \left\| G_{\rho_{n-1}}(t)\rho_n(t) + \frac{\sigma^2}{2}\rho_{n_x}(t) \right\|^2_2 dt$$

$$\leq 2\int_0^T \left\| G_{\rho_{n-1}}(t) \right\|^2_\infty \|\rho_n(t)\|^2_2 \, dt$$

$$+ \sigma^2 \int_0^T \left\| \rho_{n_x}(t) \right\|_2^2 dt$$

$$\leq 2R^2 \int_0^T \left\| \rho_n(t) \right\|_2^2 dt$$

$$+ \sigma^2 \int_0^T \left\| \rho_{n_x}(t) \right\|_2^2 dt$$

$$\leq C(T) \left\| \rho_0 \right\|_2^2, \tag{65}$$

where in the last step we used Proposition 3.3. Therefore, we have the uniform estimate

$$\left\| \rho_n \right\|_{L^\infty\left(0,T; L^2_{per}(U)\right)} + \left\| \rho_n \right\|_{L^2\left(0,T; H^1_{per}(U)\right)} + \left\| \rho_{n_t} \right\|_{L^2\left(0,T; H^{-1}_{per}(U)\right)} \leq C(T) \left\| \rho_0 \right\|_2. \tag{66}$$

Hence, $\rho \in L^2\left(0, T; H^1_{per}(U)\right) \cap L^\infty\left(0, T; L^2_{per}(U)\right)$, with $\rho_t \in L^2\left(0, T; H^{-1}_{per}(U)\right)$ and they satisfy the same estimate (66). Furthermore, there exists $\left\{ \rho_{n_k} : k \geq 1 \right\}$ such that

$$\begin{cases} \rho_{n_k} \rightharpoonup \rho & \text{in } L^2\left(0, T; H^1_{per}(U)\right), \\ \rho_{n_{k_t}} \rightharpoonup \rho_t & \text{in } L^2\left(0, T; H^{-1}_{per}(U)\right). \end{cases} \qquad \square \tag{67}$$

Following [41], we can deduce from Lemma 3.7 the following result:

Theorem 3.8. *Suppose* $\rho \in L^2\left(0, T; H^1_{per}(U)\right)$ *with* $\rho_t \in L^2\left(0, T; H^{-1}_{per}(U)\right)$, *then* $\rho \in C\left(0, T; L^2_{per}(U)\right)$ *up to a set of measure zero. Further, the mapping*

$$t \mapsto \left\| \rho(t) \right\|_2^2 \tag{68}$$

is absolutely continuous, with

$$\frac{d}{dt} \left\| \rho(t) \right\|_2^2 = 2 \left\langle \rho_t(t), \rho(t) \right\rangle, \tag{69}$$

for a.e. $0 \leq t \leq T$. *Here,* $\langle , \rangle$ *denotes the pairing between* H^{-1}_{per} *and* H^1_{per}.

Proof. The proof is identical to the proof in Evans [41], Section 5.9, Theorem 3. The only difference here is that we are considering H^1_{per} and H^{-1}_{per}, instead of H^1_0 and H^{-1}. Since periodic conditions still guarantee integration by parts without extra terms, all proofs follow through. $\square$

Now, we are ready to prove the existence of a weak solution to equation (3).

Definition 3.9. We say that $\rho \in L^2\left(0, T; H^1_{per}(U)\right) \cap L^\infty\left(0, T; L^2_{per}(U)\right)$ with $\rho_t \in L^2\left(0, T; H^{-1}_{per}(U)\right)$ is a *weak solution* of equation (3) if for every $\eta \in L^2\left(0, T; H^1_{per}(U)\right)$,

$$\int_0^T \langle \rho_t(t), \eta(t) \rangle \, dt + \int_0^T \int_U \left(\frac{\sigma^2}{2} \rho_x \eta_x + \rho G_\rho \eta_x \right) dx dt = 0, \tag{70}$$

and $\rho(0) = \rho_0$. Note that since $\rho \in C\left(0, T; L^2_{per}(U)\right)$ (Theorem 3.8), the last condition makes sense as an initial condition.

Theorem 3.10. *(Existence and uniqueness) Let $\rho_0 \in C^\infty_{per}(U)$, $\rho_0 \geq 0$ and $\int_U \rho_0(x)\, dx = 1$. Then, there exists a unique weak solution $\rho \in L^\infty\left(0, T; L^2_{per}(U)\right) \cap L^2\left(0, T; H^1_{per}(U)\right)$, with $\rho_t \in L^2\left(0, T; H^{-1}_{per}(U)\right)$, to equation (3) with the estimate*

$$\|\rho\|_{L^\infty\left(0,T;L^2_{per}(U)\right)} + \|\rho\|_{L^2\left(0,T;H^1_{per}(U)\right)} + \|\rho_t\|_{L^2\left(0,T;H^{-1}_{per}(U)\right)} \leq C(T) \|\rho_0\|_2 .$$

Proof. For each $\eta \in L^2\left(0, T; H^1(U)\right)$, we multiply equation (35) (with $n = n_k$) by η and integrate over U_T to obtain

$$\int_0^T \langle \rho_{n_{k_t}}(t), \eta(t) \rangle \, dt + \frac{\sigma^2}{2} \int_0^T \int_U \rho_{n_{k_x}} \eta_x dx dt + \int_0^T \int_U \eta_x \rho_{n_k} G_{\rho_{n_k-1}} dx dt = 0. \tag{71}$$

There are no boundary terms due to periodic boundary conditions. Now,

$$\int_0^T \int_U \eta_x \rho_{n_k} G_{\rho_{n_k-1}} dx dt = \int_0^T \int_U \eta_x (\rho_{n_k} - \rho) G_{\rho_{n_k-1}} dx dt$$

$$+ \int_0^T \int_U \eta_x \rho G_{\left(\rho_{n_k-1}-\rho\right)} dx dt$$

$$+ \int_0^T \int_U \eta_x \rho G_\rho dx dt. \tag{72}$$

We know from Lemma 3.7 that $\rho_{n_k} \rightharpoonup \rho$ in $L^2\left(0, T; H^1_{per}(U)\right) \subset L^2\left(0, T; L^2_{per}(U)\right)$. Moreover, $G_{\rho_{n_k-1}}$ is uniformly bounded so that $\eta_x G_{\rho_{n_k-1}} \in L^2\left(0, T; L^2_{per}(U)\right)$. Thus,

$$\int_0^T \int_U \eta_x \left(\rho_{n_k} - \rho\right) G_{\rho_{n_k-1}} dx dt \to 0. \tag{73}$$

Also,

$$\int_0^T \int_U \eta_x \rho G_{\left(\rho_{n_k-1} - \rho\right)} dx dt \le \|\rho\|_{L^\infty\left(0,T;L^2_{per}(U)\right)} \|\eta_x\|_{L^2\left(0,T;L^2_{per}(U)\right)}$$

$$\left\| G_{\left(\rho_{n_k-1} - \rho\right)} \right\|_{L^2\left(0,T;L^2_{per}(U)\right)}$$

$$\le C\left(T\right) \|\rho_0\|_2 \|\eta\|_{L^2\left(0,T;H^1_{per}(U)\right)}$$

$$\left(\int_0^T \|\rho_{n_k-1}\left(t\right) - \rho\left(t\right)\|_1^2 dt\right)^{1/2}. \tag{74}$$

But $\left\|\rho_{n_k-1}\left(t\right) - \rho\left(t\right)\right\|_1 \le 2R$. Hence, by the strong convergence result in Lemma 3.5, we have

$$\int_0^T \|\rho_{n_k-1}\left(t\right) - \rho\left(t\right)\|_1^2 dt \le 2R \|\rho_{n_k-1}\left(t\right) - \rho\left(t\right)\|_{L^1\left(0,T;L^1(U)\right)}$$

$$\to 0, \tag{75}$$

and thus

$$\int_0^T \int_U \eta_x \rho G_{\left(\rho_{n_k-1} - \rho\right)} dx dt \to 0. \tag{76}$$

Combining (72), (73) and (76), we have

$$\int_0^T \int_U \eta_x \rho_{n_k} G_{\rho_{n_k-1}} dx dt \to \int_0^T \int_U \eta_x \rho G_\rho dx dt. \tag{77}$$

By the weak convergence results established in Lemma 3.7, we also have

$$\int_0^T \left\langle \rho_{n_{k_t}}\left(t\right), \eta\left(t\right)\right\rangle dt \to \int_0^T \left\langle \rho_t\left(t\right), \eta\left(t\right)\right\rangle dt,$$

$$\int_0^T \int_U \rho_{n_{k_x}} \eta_x dx dt \to \int_0^T \int_U \rho_x \eta_x dx dt. \tag{78}$$

Putting together (77) and (78), we obtain in the limit $k \to \infty$,

$$\int_0^T \langle \rho_t(t), \eta(t) \rangle \, dt + \int_0^T \int_U \left(\frac{\sigma^2}{2} \rho_x \eta_x + \rho G_\rho \eta_x \right) dx dt = 0, \tag{79}$$

for every $\eta \in L^2 \left(0, T; H^1_{per}(U)\right)$.

Finally, we have to show that $\rho(0) = \rho_0$. Pick some $\eta \in C^1 \left(0, T; H^1_{per}(U)\right)$ with $\eta(T) = 0$. Then, we have from (79) that

$$-\int_0^T \langle \rho(t), \eta_t(t) \rangle \, dt + \int_0^T \int_U \left(\frac{\sigma^2}{2} \rho_x \eta_x + \rho G_\rho \eta_x \right) dx dt = (\rho(0), \eta(0)). \tag{80}$$

Similarly, we also have

$$-\int_0^T \langle \rho_{n_k}(t), \eta_t(t) \rangle \, dt + \int_0^T \int_U \left(\frac{\sigma^2}{2} \rho_{n_{k_x}} \eta_x + \rho_{n_k} G_{\rho_{n_{k-1}}} \eta_x \right) dx dt$$

$$= (\rho_0, \eta(0)). \tag{81}$$

Where we have used the fact that $\rho_{n_k}(0) = \rho_0$ for all k. Taking the limit $k \to \infty$ and comparing (80) and (81), we have

$$(\rho(0), \eta(0)) = (\rho_0, \eta(0)). \tag{82}$$

Since η is arbitrary, we conclude that $\rho(0) = \rho_0$. This completes the proof of the existence of a weak solution.

Now, we prove its uniqueness. Let ρ_1 and ρ_2 be weak solutions to (3) and set $\xi = \rho_1 - \rho_2$. Then, for every $\eta \in L^2 \left(0, T; H^1_{per}(U)\right)$, we have

$$\int_0^T \langle \xi_t(t), \eta(t) \rangle \, dt + \frac{\sigma^2}{2} \int_0^T \int_U \xi_x \eta_x dx dt + \int_0^T \int_U \left(\rho_1 G_{\rho_1} - \rho_2 G_{\rho_2} \right) \eta_x dx dt = 0. \tag{83}$$

Adding and subtracting $\int_0^T \int_U \rho_2 G_{\rho_1} \eta_x dx dt$, we obtain

$$\int_0^T \langle \xi_t(t), \eta(t) \rangle \, dt + \frac{\sigma^2}{2} \int_0^T \int_U \xi_x \eta_x dx dt = -\int_0^T \int_U \xi G_{\rho_1} \eta_x dx dt$$

$$-\int_0^T \int_U \rho_2 G_\xi \eta_x dx dt. \tag{84}$$

But,

$$
\left| \int\limits_0^T \int\limits_U \xi G_{\rho_1} \eta_x \, dx \, dt \right| \leq \| G_{\rho_1} \|_{L^\infty \left(0,T; L^\infty_{per}(U) \right)}
$$

$$
\times \| \eta_x \|_{L^2 \left(0,T; L^2_{per}(U) \right)} \| \xi \|_{L^2 \left(0,T; L^2_{per}(U) \right)}
$$

$$
\leq R \, \| \eta_x \|_{L^2 \left(0,T; L^2_{per}(U) \right)} \| \xi \|_{L^2 \left(0,T; L^2_{per}(U) \right)}
$$

$$
\leq \frac{\sigma^2}{4} \| \eta_x \|^2_{L^2 \left(0,T; L^2_{per}(U) \right)}
$$

$$
+ C \, \| \xi \|^2_{L^2 \left(0,T; L^2(U) \right)},
\tag{85}
$$

and

$$
\left| \int\limits_0^T \int\limits_U \rho_2 G_\xi \eta_x \, dx \, dt \right| \leq \| \rho_2 \|_{L^\infty \left(0,T; L^2_{per}(U) \right)}
$$

$$
\times \| G_\xi \|_{L^2 \left(0,T; L^2_{per}(U) \right)} \| \eta_x \|_{L^2 \left(0,T; L^2_{per}(U) \right)}
$$

$$
\leq C \, (T) \, \| \rho_0 \|_2 \, \| \eta_x \|_{L^2 \left(0,T; L^2_{per}(U) \right)} \| \xi \|_{L^2 \left(0,T; L^2_{per}(U) \right)}
$$

$$
\leq \frac{\sigma^2}{4} \| \eta_x \|^2_{L^2 \left(0,T; L^2_{per}(U) \right)}
$$

$$
+ C \, (T) \, \| \rho_0 \|^2_2 \, \| \xi \|^2_{L^2 \left(0,T; L^2_{per}(U) \right)},
\tag{86}
$$

so that

$$
\int\limits_0^T \langle \xi_t \, (t) \, , \eta \, (t) \rangle \, dt + \frac{\sigma^2}{2} \int\limits_0^T \int\limits_U \xi_x \eta_x \, dx \, dt
$$

$$
\leq \frac{\sigma^2}{2} \| \eta_x \|^2_{L^2(0,T; L^2(U))}
$$

$$
+ \left(C_1 \, (T) + C_2 \, (T) \, \| \rho_0 \|^2_2 \right) \| \xi \|^2_{L^2(0,T; L^2(U))}.
\tag{87}
$$

Now, set $\eta = \xi$, and use Theorem 3.8, we have

$$
\int\limits_0^T \frac{1}{2} \frac{d}{dt} \| \xi \, (t) \|^2_2 \, dt \leq \left(C_1 \, (T) + C_2 \, (T) \, \| \rho_0 \|^2_2 \right) \| \xi \|^2_{L^2(0,T; L^2(U))}
$$

$$
= \left(C_1 \, (T) + C_2 \, (T) \, \| \rho_0 \|^2_2 \right) \int\limits_0^T \| \xi \, (t) \|^2_2 \, dt.
\tag{88}
$$

Since this holds for all T, we must have

$$\frac{d}{dt} \|\xi(t)\|_2^2 \leq \left(C_1(T) + C_2(T) \|\rho_0\|_2^2 \right) \|\xi(t)\|_2^2, \tag{89}$$

and hence

$$\|\xi(t)\|_2 \leq \left(C_1(T) + C_2(T) \|\rho_0\|_2^2 \right) \|\xi(0)\|_2, \tag{90}$$

for a.e. $0 \leq t \leq T$. But $\|\xi(0)\|_2^2 = \|\rho_0 - \rho_0\|_2^2 = 0$ and the ρ's are continuous in time, we have

$$\|\rho_1(t) - \rho_2(t)\|_2 = 0 \tag{91}$$

for all $0 \leq t \leq T$.

Finally, the energy estimate is from Lemma 3.7. □

Remark 3.11. The strong convergence result (Lemma 3.5) is important here because without it, we could not have concluded that expression (75) converges to 0, because it involves a different subsequence ρ_{n_k-1}. Strong convergence ensures that all subsequences converge in $L^1\left(0, T; L_{per}^1(U)\right)$.

Throughout this section we assumed that the initial condition is smooth, i.e. $\rho_0 \in C_{per}^\infty(U)$. We can in fact relax this condition to $\rho_0 \in L_{per}^2(U)$ by mollifying the initial data.

Theorem 3.12. *(Existence and uniqueness with relaxed regularity assumption on the initial condition) Let $\rho_0 \geq 0$, $\int_U \rho_0(x)\,dx = 1$ and $\rho_0 \in L_{per}^2(U)$. Then, there exists a unique weak solution $\rho \in L^\infty\left(0, T; L_{per}^2(U)\right) \cap L^2\left(0, T; H_{per}^1(U)\right)$, with $\rho_t \in L^2\left(0, T; H_{per}^{-1}(U)\right)$ to equation (3) with the estimate*

$$\|\rho\|_{L^\infty\left(0,T;L_{per}^2(U)\right)} + \|\rho\|_{L^2\left(0,T;H_{per}^1(U)\right)} + \|\rho_t\|_{L^2\left(0,T;H_{per}^{-1}(U)\right)} \leq C(T) \|\rho_0\|_2.$$

Proof. Let $\epsilon > 0$ and consider the modified problem

$$\begin{cases} \rho_{n_t}^\epsilon - \dfrac{\sigma^2}{2} \rho_{n_{xx}}^\epsilon = \left(\rho_n^\epsilon G_{\rho_{n-1}^\epsilon} \right)_x & \text{in } U_T \\ \rho_n^\epsilon(\ell, \cdot) = \rho_n^\epsilon(-\ell, \cdot) & \text{on } \partial U \times [0, T] \\ \rho_n^\epsilon = \rho_0^\epsilon & \text{on } U \times \{t = 0\} \end{cases} \tag{92}$$

where

$$\rho_0^\epsilon(x) := \int_U j_\epsilon(x - y) \rho_0(y)\,dy, \tag{93}$$

and $j_\epsilon(x) = \epsilon^{-1} j\left(\epsilon^{-1} x\right)$. Here, j is a standard positive mollifier with compact support on U and $\int_U j_\epsilon(x)\,dx = 1$.

With mollification, ρ_0^ϵ is now smooth and we can apply Theorem 3.10 to conclude that there exists a unique weak solution $\rho^\epsilon \in L^\infty \left(0, T; L^2_{per}(U)\right) \cap L^2 \left(0, T; H^1_{per}(U)\right)$, with $\rho_t^\epsilon \in L^2 \left(0, T; H^{-1}_{per}(U)\right)$ to equation (92) with the estimate

$$\left\| \rho^\epsilon \right\|_{L^\infty \left(0,T; L^2_{per}(U)\right)} + \left\| \rho^\epsilon \right\|_{L^2 \left(0,T; H^1_{per}(U)\right)} + \left\| \rho_t^\epsilon \right\|_{L^2 \left(0,T; H^{-1}_{per}(U)\right)} \leq C(T) \left\| \rho_0^\epsilon \right\|_2. \qquad (94)$$

But for all ϵ, we have $\left\| \rho_0^\epsilon \right\|_2 \leq \left\| \rho_0 \right\|_2$. Hence, there exists $\rho \in L^\infty \left(0, T; L^2_{per}(U)\right) \cap L^2 \left(0, T; H^1_{per}(U)\right)$, with $\rho_t \in L^2 \left(0, T; H^{-1}_{per}(U)\right)$, satisfying

$$\left\| \rho \right\|_{L^\infty \left(0,T; L^2_{per}(U)\right)} + \left\| \rho \right\|_{L^2 \left(0,T; H^1_{per}(U)\right)} + \left\| \rho_t \right\|_{L^2 \left(0,T; H^{-1}_{per}(U)\right)} \leq C(T) \left\| \rho_0 \right\|_2, \qquad (95)$$

and a sequence $\{\epsilon_k\}$, with $\epsilon_k \to 0$, such that

$$\begin{cases} \rho^{\epsilon_k} \rightharpoonup \rho & \text{in } L^2 \left(0, T; H^1_{per}(U)\right), \\ \rho_t^{\epsilon_k} \rightharpoonup \rho_t & \text{in } L^2 \left(0, T; H^{-1}_{per}(U)\right), \end{cases} \qquad (96)$$

as $k \to \infty$. We now show that ρ is in fact a weak solution to (3). Since each ρ^{ϵ_k} solves the weak formulation of (92) (albeit with different initial data), we have

$$\int\limits_0^T \left\langle \rho^{\epsilon_k}(t), \eta(t) \right\rangle dt + \frac{\sigma^2}{2} \int\limits_0^T \int\limits_U \rho^{\epsilon_k} \eta_x \, dx dt + \int\limits_0^T \int\limits_U \eta_x \rho^{\epsilon_k} G_{\rho^{\epsilon_k}} \, dx dt = 0. \qquad (97)$$

Using (96), we can replace ρ^{ϵ_k} by ρ in the first two integrals above in the limit $k \to \infty$. Moreover, as in (72), we write the last integral as

$$\int\limits_0^T \int\limits_U \eta_x \rho^{\epsilon_k} G_{\rho^{\epsilon_k}} \, dx dt = \int\limits_0^T \int\limits_U \eta_x \left(\rho^{\epsilon_k} - \rho \right) G_{\rho^{\epsilon_k}} \, dx dt$$

$$+ \int\limits_0^T \int\limits_U \eta_x \rho G_{(\rho^{\epsilon_k} - \rho)} \, dx dt$$

$$+ \int\limits_0^T \int\limits_U \eta_x \rho G_\rho \, dx dt. \qquad (98)$$

Since $\left\| G_{\rho^{\epsilon_k}}(t) \right\|_\infty \leq R \left\| \rho^{\epsilon_k}(t) \right\|_1 \leq R \left\| \rho_0^{\epsilon_k} \right\|_1 = R \left\| \rho_0 \right\|_1 = R$, we have $\eta_x G_{\rho^{\epsilon_k}} \in L^2 \left(0, T; L^2_{per}(U)\right)$ and hence

$$\int_0^T \int_U \eta_x \left(\rho^{\epsilon_k} - \rho \right) G_{\rho^{\epsilon_k}} dxdt \to 0. \tag{99}$$

Next, we can write

$$\int_0^T \int_U \eta_x \rho G_{(\rho^{\epsilon_k} - \rho)} dxdt = \int_0^T \int_U \eta_x \rho \left[\int_{x-R}^{x+R} \left(\rho^{\epsilon_k} - \rho \right) (x - y) dy \right] dxdt$$

$$= \int_0^T \int_U h \left(\rho^{\epsilon_k} - \rho \right) dydt, \tag{100}$$

where we have defined

$$h(y, t) := \int_{y-R}^{y+R} \eta_x (x, t) \rho (x, t) (x - y) dx. \tag{101}$$

Clearly, $\| h(t) \|_\infty \le R \| \eta_x (t) \|_2 \| \rho (t) \|_2$ so that in particular, $h \in L^2 \left(0, T, L^2_{per} (U) \right)$ and from (100) we obtain

$$\int_0^T \int_U \eta_x \rho G_{(\rho^{\epsilon_k} - \rho)} dxdt \to 0. \tag{102}$$

Thus, we have shown that ρ satisfies

$$\int_0^T \langle \rho(t), \eta(t) \rangle \, dt + \frac{\sigma^2}{2} \int_0^T \int_U \rho \eta_x dxdt + \int_0^T \int_U \eta_x \rho G_\rho dxdt = 0. \tag{103}$$

To show that $\rho(0) = \rho_0$, we again take $\eta \in C^1 \left(0, T; H^1_{per} (U) \right)$ with $\eta(T) = 0$. Since $\rho_0^{\epsilon_k} \to \rho_0$ uniformly, we have (cf. expressions (80) and (81))

$$(\rho(0), \eta(0)) = (\rho_0, \eta(0)). \tag{104}$$

Since η is arbitrary, we have $\rho(0) = \rho_0$. The uniqueness follows from exactly the same argument in the proof of Theorem 3.10 and we omit writing it again here. □

4. Higher regularity

In this section, we prove improved regularity of the weak solution to (3). This allows us to put the results in Section 2 on a rigorous footing. As in the previous section, we always mollify ρ_0 by j_ϵ so that the resulting evolution equations (35) admit smooth solutions. This allows us to differentiate the equation as many times as required, and we take the limit $\epsilon \to 0$ at the end. For simplicity of notation, we drop the ϵ superscripts on ρ_n and implicitly assume that we perform the limit at the end.

First, we prove a useful estimate.

Proposition 4.1. *Let* $u, v \in C^\infty (U)$. *Then for* $k \geq 2$ *we have the estimate*

$$\|uG_v\|_{H_{per}^k (U)} \leq C \|u\|_{H_{per}^k (U)} \|v\|_{H_{per}^{k-1} (U)} . \tag{105}$$

Proof. We have

$$\|uG_v\|_{H_{per}^k (U)}^2 \leq C \left(\|uG_v\|_2^2 + \left\| (uG_v)^{(k)} \right\|_2^2 \right) ,$$

where $(\cdot)^{(k)}$ denotes the kth derivative with respect to x. Applying the Leibniz rule, we have

$$(uG_v)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} u^{(k-i)} (G_v)^{(i)} . \tag{106}$$

But,

$$(G_v)^{(i)} (x) = \begin{cases} G_v (x) & i = 0 \\ -R \left[v (x + R) + v (x - R) \right] + \int_{x-R}^{x+R} v (y) \, dy & i = 1 \\ -R \left[v^{(i-1)} (x + R) + v^{(i-1)} (x - R) \right] & \\ \quad + v^{(i-2)} (x + R) - v^{(i-2)} (x - R) & i \geq 2 \end{cases} \tag{107}$$

Hence we have the bound

$$\left\| (uG_v)^{(k)} \right\|_2^2 \leq C_0 \left\| u^{(k)} \right\|_2^2 \|G_v\|_\infty^2$$

$$+ \sum_{i=1}^{k} C_i \left\| u^{(k-i)} \right\|_\infty^2 \left\| (G_v)^{(i)} \right\|_2^2$$

$$\leq C_0 \|u\|_{H_{per}^k (U)}^2 \|v\|_2^2$$

$$+ \sum_{i=1}^{k} C_i \left\| u^{(k-i)} \right\|_{H_{per}^1 (U)}^2 \left\| (G_v)^{(i)} \right\|_2^2 . \tag{108}$$

For $i \geq 2$,

$$\left\| (G_v)^{(i)} \right\|_2^2 \le C \left(\left\| v^{(i-1)} \right\|_2^2 + \left\| v^{(i-2)} \right\|_2^2 \right)$$

$$\le C \left\| v \right\|_{H_{per}^{i-1}(U)}^2, \tag{109}$$

and for $i = 1$,

$$\left\| (G_v)^{(i)} \right\|_2^2 \le C \left(\left\| v \right\|_2^2 + \left\| v \right\|_1^2 \right)$$

$$\le C \left\| v \right\|_2^2. \tag{110}$$

Keeping only the highest Sobolev norms, we have

$$\left\| (uG_v)^{(k)} \right\|_2^2 \le C_0 \left\| u \right\|_{H_{per}^k(U)}^2 \left\| v \right\|_2^2 + C_1 \left\| u \right\|_{H_{per}^k(U)}^2 \left\| v \right\|_{H_{per}^{k-1}(U)}^2$$

$$\le C \left\| u \right\|_{H_{per}^k(U)}^2 \left\| v \right\|_{H_{per}^{k-1}(U)}^2. \qquad \square \tag{111}$$

Now, we assume that $\rho_0 \in H_{per}^k$ for some $k \ge 0$ and prove the corresponding regularity of ρ.

Theorem 4.2. *(Improved regularity) Let $k \ge 0$ and suppose $\rho_0 \in H_{per}^k(U)$ with $\rho_0 \ge 0$ and $\int_U \rho_0(x)\,dx = 1$. Then the unique solution to (3) satisfies*

$$\rho \in L^2 \left(0, T; H_{per}^{k+1}(U) \right) \cap L^\infty \left(0, T; H_{per}^k(U) \right),$$

with the estimate

$$\left\| \rho \right\|_{L^2 \left(0,T; H_{per}^{k+1}(U) \right)} + \left\| \rho \right\|_{L^\infty \left(0,T; H_{per}^k(U) \right)} \le C(\rho_0; k, T),$$

where

$$C(\rho_0; k, T) := C(T) \left(\sum_{i=0}^{k} \left\| \rho_0 \right\|_{H_{per}^{k-i}(U)}^{2^{i+1}} \right)^{1/2}.$$

Proof. We prove the statements by proving uniform estimates on ρ_n by induction on k. The base case $k = 0$ is provided in Proposition 3.3. The $k = 1$ case is Proposition 3.4. Suppose for some $k \ge 1$,

$$\left\| \rho_n \right\|_{L^2 \left(0,T; H_{per}^{k+1}(U) \right)} + \left\| \rho_n \right\|_{L^\infty \left(0,T; H_{per}^k(U) \right)} \le C(\rho_0; k, T), \tag{112}$$

for all n. We differentiate equation (35) k times with respect to x, multiply it by $-\partial_x^{k+2} \rho_n$ and integrate over U to get

$$\frac{1}{2}\frac{d}{dt}\left\|\partial_x^{k+1}\rho_n(t)\right\|_2^2 + \frac{\sigma^2}{2}\left\|\partial_x^{k+2}\rho_n(t)\right\|_2^2$$

$$\leq \int_U \left|\partial_x^{k+1}\rho_n(t)\,\partial_x^{k+1}\left(\rho_n G_{\rho_{n-1}}\right)(t)\right|dx$$

$$\leq \frac{\sigma^2}{4}\left\|\partial_x^{k+2}\rho_n(t)\right\|_2^2 + C\left\|\partial_x^{k+1}\left(\rho_n G_{\rho_{n-1}}\right)(t)\right\|_2^2. \tag{113}$$

Using Proposition 4.1 with $u = \rho_n(t)$ and $v = \rho_{n-1}(t)$, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\partial_x^{k+1}\rho_n(t)\right\|_2^2 + \frac{\sigma^2}{4}\left\|\partial_x^{k+2}\rho_n(t)\right\|_2^2$$

$$\leq C\left(\|\rho_n(t)\|_{H_{per}^{k+1}(U)}^2\,\|\rho_{n-1}(t)\|_{H_{per}^k(U)}^2\right) \tag{114}$$

Integrating over time, we get

$$\sup_{0\leq t\leq T}\|\rho_n(t)\|_{H_{per}^{k+1}(U)}^2 + \|\rho_n\|_{L^2\left(0,T;H_{per}^{k+2}(U)\right)}^2$$

$$\leq \|\rho_0\|_{H_{per}^{k+1}(U)}^2$$

$$\quad + C\left(\|\rho_{n-1}\|_{L^\infty\left(0,T;H_{per}^k(U)\right)}^2\,\|\rho_n\|_{L^2\left(0,T;H_{per}^{k+1}(U)\right)}^2\right)$$

$$\leq \|\rho_0\|_{H_{per}^{k+1}(U)}^2 + [C(\rho_0;k,T)]^4$$

$$\leq C(\rho_0;k+1,T)^2 \tag{115}$$

This completes the induction. Taking limits, we obtain

$$\rho \in L^2\left(0,T;H_{per}^{k+2}(U)\right)\cap L^\infty\left(0,T;H_{per}^{k+1}(U)\right), \tag{116}$$

with the estimate

$$\|\rho\|_{L^2\left(0,T;H_{per}^{k+2}(U)\right)} + \|\rho\|_{L^\infty\left(0,T;H_{per}^{k+1}(U)\right)} \leq C(\rho_0;k+1,T). \qquad \square \tag{117}$$

So far we have only considered regularity in space. The same can also be done in the time domain.

Theorem 4.3. *(Improved regularity) Let $k \geq 0$ and suppose $\rho_0 \in H_{per}^{2k}(U)$ with $\rho_0 \geq 0$ and $\int_U \rho_0(x)\,dx = 1$. Then,*
(i) For every $0 \leq m \leq k$, the unique solution to (3) satisfies

$$\frac{d^m\rho}{dt^m} \in L^2\left(0,T;H_{per}^{2k-2m+1}(U)\right)\cap L^\infty\left(0,T;H_{per}^{2k-2m}(U)\right),$$

with the estimate

$$\sum_{m=0}^{k} \left(\left\| \frac{d^m \rho}{dt^m} \right\|_{L^2\left(0,T;H_{per}^{2k-2m+1}(U)\right)} + \left\| \frac{d^m \rho}{dt^m} \right\|_{L^\infty\left(0,T;H_{per}^{2k-2m}(U)\right)} \right) \leq D\left(\rho_0; k, T\right),$$

where

$$D\left(\rho_0; k, T\right) := \left(\sum_{j=0}^{k} C\left(\rho_0; 2k, T\right)^{2^{j+1}} \right)^{1/2}.$$

(ii) Moreover,

$$\frac{d^{k+1}\rho}{dt^{k+1}} \in L^2\left(0, T; H_{per}^{-1}(U)\right),$$

with the estimate

$$\left\| \frac{d^{k+1}\rho}{dt^{k+1}} \right\|_{L^2\left(0,T;H_{per}^{-1}(U)\right)} \leq D\left(\rho_0; k, T\right).$$

Proof. Let us prove that for all $M \leq k$,

$$\sum_{m=0}^{M} \left(\left\| \frac{d^m \rho_n}{dt^m} \right\|_{L^2\left(0,T;H_{per}^{2k-2m+1}(U)\right)} + \left\| \frac{d^m \rho_n}{dt^m} \right\|_{L^\infty\left(0,T;H_{per}^{2k-2m}(U)\right)} \right)$$
$$\leq C\left(\rho_0; k, M, T\right), \tag{118}$$

where we have defined

$$C\left(\rho_0; k, M, T\right) := \left(\sum_{j=0}^{M} C\left(\rho_0; 2k, T\right)^{2^{j+1}} \right)^{1/2}. \tag{119}$$

This is done by induction on M up to k. The case $M = 0$ is Theorem 4.2. Suppose we have for some $0 \leq M < k$ the estimate (118). Differentiating equation (35) M times with respect to t and using the Leibniz rule, we have

$$\rho_n^{(M+1)} = \frac{\sigma^2}{2} \rho_{n_{xx}}^{(M)} + \left(\rho_n G_{\rho_{n-1}} \right)_x^{(M)}$$
$$= \frac{\sigma^2}{2} \rho_{n_{xx}}^{(M)} + C \sum_{m=0}^{M} \left(\rho_n^{(m)} G_{\rho_{n-1}^{(M-m)}} \right)_x, \tag{120}$$

where we used the shorthand $\rho_n^{(m)} := \partial^m \rho_n / \partial t^m$. Thus, we have

$$\left\| \rho_n^{(M+1)}(t) \right\|_{H_{per}^{2k-2M-1}(U)}^2 \leq C_1 \left\| \rho_n^{(M)}(t) \right\|_{H_{per}^{2k-2M+1}(U)}^2$$

$$+ C_2 \sum_{m=0}^{M} \left\| \rho_n^{(m)}(t) \, G_{\rho_{n-1}^{(M-m)}}(t) \right\|_{H_{per}^{2k-2M}(U)}^2 . \tag{121}$$

Using [Proposition 4.1] with $u = \rho_n^{(m)}(t)$ and $v = \rho_{n-1}^{(M-m)}(t)$, we have

$$\left\| \rho_n^{(M+1)}(t) \right\|_{H_{per}^{2k-2M-1}(U)}^2$$

$$\leq C_1 \left\| \rho_n^{(M)}(t) \right\|_{H_{per}^{2k-2M+1}(U)}^2$$

$$+ C_2 \sum_{m=0}^{M} \left\| \rho_n^{(m)}(t) \, G_{\rho_{n-1}^{(M-m)}}(t) \right\|_{H_{per}^{2k-2M}(U)}^2$$

$$\leq C_1 \left\| \rho_n^{(M)}(t) \right\|_{H_{per}^{2k-2M+1}(U)}^2$$

$$+ C_2 \sum_{m=0}^{M} \left\| \rho_n^{(m)}(t) \right\|_{H_{per}^{2k-2M}(U)}^2 \left\| \rho_{n-1}^{(M-m)}(t) \right\|_{H_{per}^{2k-2M-1}(U)}^2$$

$$\leq C_1 \left\| \rho_n^{(M)}(t) \right\|_{H_{per}^{2k-2M+1}(U)}^2$$

$$+ C_2 \sum_{m=0}^{M} \left\| \rho_n^{(m)}(t) \right\|_{H_{per}^{2k-2m+1}(U)}^2 \left\| \rho_{n-1}^{(M-m)}(t) \right\|_{H_{per}^{2k-2(M-m)}(U)}^2 . \tag{122}$$

Integrating over time then gives

$$\left\| \rho_n^{(M+1)} \right\|_{L^2\left(0,T;H_{per}^{2k-2M-1}(U)\right)}^2$$

$$\leq C_1 \left\| \rho_n^{(M)} \right\|_{L^2\left(0,T;H_{per}^{2k-2M+1}(U)\right)}^2$$

$$+ C_2 \sum_{m=0}^{M} \left\| \rho_n^{(m)} \right\|_{L^2\left(0,T;H_{per}^{2k-2m+1}(U)\right)}^2 \left\| \rho_{n-1}^{(M-m)} \right\|_{L^\infty\left(0,T;H_{per}^{2k-2(M-m)}(U)\right)}^2 . \tag{123}$$

Since $0 \leq m, M - m \leq M$, we can apply the inductive hypothesis (118) to conclude that

$$\left\| \rho_n^{(M+1)} \right\|_{L^2\left(0,T;H_{per}^{2k-2M-1}(U)\right)}^2 \leq C\left(\rho_0; k, M, T\right)^2 + C\left(\rho_0; k, M, T\right)^4$$

$$\leq C\left(\rho_0; k, M+1, T\right)^2 . \tag{124}$$

Similarly,

$$\left\| \rho_n^{(M+1)} \right\|_{L^\infty\left(0,T;H_{per}^{2k-2M-2}(U)\right)}^2$$

$$\leq C_1 \left\| \rho_n^{(M)} \right\|_{L^\infty\left(0,T;H_{per}^{2k-2M}(U)\right)}^2$$

$$+ C_2 \sum_{m=0}^M \left\| \rho_n^{(m)} \right\|_{L^\infty\left(0,T;H_{per}^{2k-2m}(U)\right)}^2 \left\| \rho_{n-1}^{(M-m)} \right\|_{L^\infty\left(0,T;H_{per}^{2k-2(M-m)}(U)\right)}^2.$$

$$\leq C\left(\rho_0;k,M+1,T\right)^2. \tag{125}$$

This completes the induction on M up to k. Putting $M = k$ into (118) and taking limits prove part (i).

To prove the second part, notice that

$$\rho_n^{(k+1)} = \left(\frac{\sigma^2}{2} \rho_{n_x}^{(k)} + C \sum_{m=0}^k \left(\rho_n^{(m)} G_{\rho_{n-1}^{(k-m)}} \right) \right)_x. \tag{126}$$

Hence,

$$\left\| \rho_n^{(k+1)} \right\|_{L^2\left(0,T;H_{per}^{-1}(U)\right)}^2$$

$$\leq \left\| \frac{\sigma^2}{2} \rho_{n_x}^{(k)} + C \sum_{m=0}^k \left(\rho_n^{(m)} G_{\rho_{n-1}^{(k-m)}} \right) \right\|_{L^2\left(0,T;L_{per}^2(U)\right)}^2$$

$$\leq C_1 \left\| \rho_n^{(k)} \right\|_{L^2\left(0,T;H_{per}^1(U)\right)}^2$$

$$+ C_2 \sum_{m=0}^k \left\| \rho_n^{(m)} \right\|_{L^\infty\left(0,T;L_{per}^2(U)\right)}^2 \left\| \rho_{n-1}^{(M-m)} \right\|_{L^2\left(0,T;L_{per}^2(U)\right)}^2$$

$$\leq D\left(\rho_0;k,T\right). \tag{127}$$

Taking limits then proves part (ii). $\quad\square$

Corollary 4.4. *Let $T > 0$ and $\rho_0 \in H_{per}^3(U)$ with $\rho_0 \geq 0$ and $\int_U \rho_0(x)\,dx = 1$. Then the unique solution to (3) satisfies*

$$\rho \in C^1\left(0,T;C_{per}^2(U)\right),$$

after possibly being redefined on a set of measure zero.

Proof. By Theorem 4.2, $\rho \in L^\infty\left(0,T;H_{per}^3(U)\right)$, i.e. $\rho_{xx} \in L^\infty\left(0,T;H_{per}^1(U)\right)$. Hence there exists a version of ρ with $\rho_{xx}(t) \in C_{per}^{0,\frac{1}{2}}(U)$, so that in particular, $\rho(t) \in C_{per}^2(U)$. Next,

using Theorem 4.3, we have $\rho_t \in L^2\left(0, T; H^1_{per}(U)\right)$ and $\rho_{tt} \in L^2\left(0, T; H^{-1}_{per}(U)\right)$, hence by Theorem 3.8 there is a version of ρ so that $\rho_t \in C\left(0, T; L^2_{per}(U)\right)$. Hence we have

$$\rho \in C^1\left(0, T; C^2_{per}(U)\right), \tag{128}$$

up to a set of measure zero. $\quad\square$

This result allows us to restate the results in Section 2 without the a priori smoothness assumption. We summarize the main results of this paper in the following:

Theorem 4.5. *Let $\rho_0 \in H^3_{per}(U)$ with $\rho_0 \geq 0$ and $\int_U \rho_0(x)\,dx = 1$. Then, there exists a unique weak solution ρ to equation (3), with*

(i) *(Regularity)* $\rho \in C^1\left(0, \infty; C^2_{per}(U)\right)$.
(ii) *(Nonnegativity)* $\rho(t) \geq 0$ *for all* $t \geq 0$.
(iii) *(Stability) Furthermore, if* $\sigma^2 > \frac{2\ell}{\pi}\left(2R + R^2/\sqrt{3}\,\ell\right)$, *then* $\rho(t) \to \frac{1}{2\ell}$ *in* L^2_{per} *exponentially as* $t \to \infty$.

Proof. Existence and uniqueness follow from Theorem 3.12. (i) follows from Corollary 4.4. Having established (i), (ii) and (iii) then follows from Corollary 2.2 and Theorem 2.3 respectively. $\quad\square$

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