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THE TOTAL s-ENERGY OF A MULTIAGENT SYSTEM*

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Abstract. We introduce the *total s-energy* of a multiagent system with time-dependent links. This provides a new analytical perspective on bidirectional agreement dynamics, which we use to bound the convergence rates of dynamical systems for synchronization, flocking, opinion dynamics, and social epistemology.

Key words. multiagent systems, agreement dynamics, consensus, convergence rate

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1. Introduction.

We introduce an analytical device for the study of multiagent agreement systems. Consider an infinite sequence of graphs, $G_0, G_1, G_2, \ldots$, each one defined on n nodes labeled $1, \ldots, n$. We assume that each graph G_t is embedded in Euclidean d-space, and we let $x_i(t) \in \mathsf{R}^d$ denote the position of node i at time t. The *total s-energy* $E(s)$ of the embedded graph sequence encodes all of the edge lengths:

$$E(s) = \sum_{t \geq 0} \sum_{(i,j) \in G_t} \|x_i(t) - x_j(t)\|_2^s \,, \tag{1}$$

where the exponent s is a real (or complex) variable. The definition generalizes both the Dirichlet form derived from the graph Laplacian and the Riesz s-energy of points on a sphere. Sometimes, variants of the total s-energy are more convenient; for example, we will use the *kinetic s-energy*,

$$K(s) = \sum_{t \geq 0} \sum_{i=1}^{n} \|x_i(t+1) - x_i(t)\|_2^s \,. \tag{2}$$

Note that these definitions make no assumptions about the *temporal network*, which is the name given to a graph sequence sharing the same node set. There is no reason to think that s-energies should even be finite, let alone useful: for example, $E(0)$ is usually infinite. In fact, it is immediate to embed a temporal network so as to make its total s-energy diverge everywhere, so one clearly needs assumptions on the embeddings. In this paper we consider the case of *multiagent agreement systems* [33], which we define in the next section. There are two kinds: bidirectional and non–bidirectional. We consider only the former type in this work. We thus assume that each G_t is undirected, meaning that if (i, j) is an edge, then so is (j, i). (The directed case is quite different and warrants a separate treatment.)

We use the total s-energy to bound the convergence rates of classical systems for opinion dynamics (section 2.1), social epistemology (section 2.2), Kuramoto synchronization (section 2.3), and bird flocking (section 2.4). We deal only with discrete-time

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dynamical systems or, as in [33], time-1 maps of continuous systems. We also improve a classic bound for the products of stochastic matrices (section 2.5).

Our proofs are algorithmic and make no use of algebraic graph theory (with a single exception for reversible systems (section 1.1), where we turn to a standard ℓ_2 argument). In particular, the proofs focus on the agents rather than on the matrices governing their dynamics. In fact, the proofs themselves can be viewed as dynamical systems which embed convergence measures directly within the interagent communication. We hope this, perhaps opaque, comment finds clarification below and that the benefits of an algorithmic approach to multiagent dynamics becomes apparent [6, 8].

1.1. Multiagent dynamics. Moreau [33] introduced a geometric framework for multiagent agreement dynamics of appealing generality. He established convergence criteria based on connectivity in both the directed and the undirected case. We seek to analyze the dynamics of bidirectional systems *without any connectivity assumptions*: specifically, our goal is to provide bounds on convergence rates that hold in *all* cases.

Bidirectional agreement systems. The one-dimensional case features all the techniques used in this paper and it is straightforward to extend our analysis to $d > 1$; we briefly mention how to do that below. For simplicity, therefore, we assume that $d = 1$. The model involves n agents located at the points $x_1(t), \ldots, x_n(t)$ in R at any time $t \geq 0$. The input consists of their positions at time $t = 0$, together with an infinite sequence $(G_t)_{t\geq 0}$ of undirected graphs over $n \geq 2$ nodes (the agents); each node has a self-loop. These graphs represent the various configurations of a communication network changing over time. The sequence need not be known ahead of time: in practice, the system will often be embedded in a closed loop, and the next G_t will be a function of the configuration at time $0, \ldots, t-1$. The strength of the model is that it makes no assumption about the generation of the temporal networks nor about their connectivity properties. In the case of directed graphs, such a level of generality precludes blanket statements about convergence; bidirectionality, on the other hand, allows such statements. The neighbors of i form the set $N_i(t) = \{\, j \,|\, (i,j) \in G_t \,\}$, which includes i. At time t, each agent i moves *anywhere* within the interval formed by its neighbors, though not too close to the boundary: formally, if $m_{i,t}$ is the minimum of $\{\, x_j(t) \,|\, j \in N_i(t) \,\}$ and $M_{i,t}$ is the maximum, then

$$(1-\rho)m_{i,t} + \rho M_{i,t} \leq x_i(t+1) \leq \rho m_{i,t} + (1-\rho)M_{i,t}, \tag{3}$$

where $0 < \rho \leq 1/2$ is the (time-independent) *agreement parameter*, fixed once and for all. All of the agents are updated in parallel at each step $t = 0, 1, 2$, etc. We conclude the presentation of bidirectional agreement systems with a few remarks.

- The model describes a *nondeterministic* dynamical system. This refers to the fact that the sequence of graphs, as well as the particular motion of the agents, are left completely arbitrary within the constraints imposed by (3): they could be decided ahead of time or, as is more common, endogenously in a closed-loop system; we give several examples below. The embedding at time 0 is provided as input and, from then on, all subsequent embeddings are generated by the system itself in abidance of rule (3). It may seem surprising at first that one can prove convergence in the presence of such high nondeterminism and without the slightest assumption about connectivity.
- Bidirectionality does not imply symmetry among neighbors. In fact, the behavior of neighboring agents may be completely different. The condition $\rho > 0$ is essential. Without it, a two-agent system with a single edge could see the agents swap places forever without ever converging. This simple example

shows that one may legally move the two agents toward each other so that their distance decreases by a factor of merely $1-2\rho$ at each step. This shows that no worst-case convergence rate can be faster than $e^{-2\rho t}$.

- There are several ways to extend the model to a higher dimension. Perhaps the easiest is to assume that agent i is positioned at $x_i(t) \in \mathsf{R}^d$ and then enforce (3) along each dimension. This is equivalent to having d one-dimensional systems sharing the same temporal network; it is the method we use in this paper. A different, coordinate-free approach stipulates that agent i may move anywhere within the convex hull $C_i(t)$ of its neighbors $\{\, x_j(t) \,|\, j \in N_i(t) \,\}$ but not too close to the boundary (Figure 1). This requires shrinking $C_i(t)$ by a factor of $1-\rho$ centrally toward a well-chosen center, for example, the *Löwner–John center* of $C_i(t)$, which is uniquely defined as the center of the minimum-volume ellipsoid that encloses $C_i(t)$ [14].

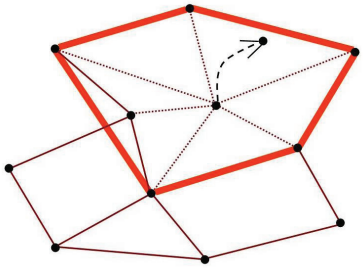

FIG. 1. *The agent can move anywhere inside the pentagon but may not touch the thick boundary.*

Much of the previous work on agreement systems has been concerned with conditions for consensus (i.e., for all agents to come together), beginning with the pioneering work of [41, 42] and then followed by [1, 2, 4, 5, 18, 19, 24, 28, 33, 34]. Bounds on the convergence rate have been obtained under various connectivity assumptions [5, 34] and for specialized closed-loop systems [7, 30]. The convergence of bidirectional agreement systems can be derived from the techniques in [18, 27, 33]. Bounding the convergence rate, however, has been left open. This is the main focus of this paper. Before stating our results in the next section, we discuss a few extensions of the model.

The fixed-agent agreement model. We can fix one agent if we so desire. By this, we mean skipping the update rule at an arbitrary agent i_0, selected ahead of time, or equivalently, directing all edges incident to i_0 toward that node. To see why fixing an agent is permissible, create the point reflection of the $n-1$ mobile agents about i_0 to create a bidirectional system of $2n-1$ agents. Figure 2 illustrates this process in two dimensions for visual clarity. We duplicate each graph G_t, with the exception of the fixed agent i_0. In this way, at time t, each edge (i,j) is given a duplicate (i',j'). Placing the origin of a Cartesian coordinate system at $x_{i_0}(0)$, we position agent i' at time 0 so that $x_{i'}(0) = -x_i(0)$, which inductively implies that $x_{i'}(t) = -x_i(t)$ for all $t \geq 0$. No edges connect the two copies of the original graphs. Every mobile agent (and its reflected copy) mimics the behavior of its counterpart in the original n-agent system while respecting (3). The fixed agent always lies at the midpoint of the smallest interval enclosing its neighbors; therefore, it does not need to move, even for the maximum value of ρ allowed, which is $1/2$. To summarize, any n-agent agreement system with one fixed agent can be simulated with a $(2n-1)$-agent bidirectional agreement system with the same value of ρ and at most twice the diameter. We apply

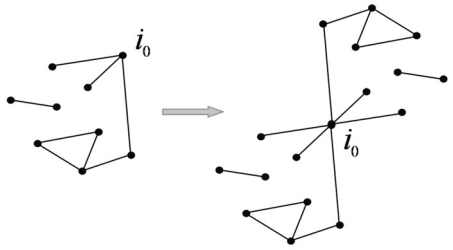

FIG. 2. *Reflecting the system about the agent i_0 that we wish to fix.*

this result to truth-seeking systems in section 2.2.

Reversible agreement systems. Assign to each agent i a time-independent *motion parameter* $q_i \geq |N_i(t)|$, and define the mass center of the agent's neighbors as

$$\mu_i(t) = \frac{1}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t)\,.$$

A *reversible agreement system* satisfies the transition

$$x_i(t+1) = x_i(t) + \frac{|N_i(t)|}{q_i}\Big(\mu_i(t) - x_i(t)\Big).$$

The agents obey the dynamics $x(t+1) = P(t)x(t)$, where

$$p_{ij}(t) = \begin{cases} 1 - (|N_i(t)| - 1)/q_i & \text{if } i = j, \\ 1/q_i & \text{if } i \neq j \in N_i(t), \\ 0 & \text{else.} \end{cases} \tag{4}$$

A quick examination of (3) shows that the dynamics, indeed, defines an agreement system with parameter $\rho = 1/\max_i q_i$. Why is it called reversible? We take note of the identity $q_i p_{ij}(t) = q_j p_{ji}(t)$. This is the standard balanced condition of a reversible Markov chain, with (q_i) in the role of the stationary distribution (up to scaling). Indeed, we easily verify that the sum $\sum_i q_i x_i(t)$ is independent of t and that a lazy random walk in a graph is a special case of a reversible agreement system. The latter is much more general, of course, since the graph can change over time. We note that, if each node has its own degree fixed over time, then moving each agent to the mass center of its neighbors satisfies reversibility and, hence, as we shall see, fast convergence. This is equivalent to setting $q_i = |N_i(t)|$.

The definition of a reversible system is simple yet somewhat contrived. Does the concept lend itself to intuition? Yes. At each step, each agent picks a set of neighbors (the graph G_t) and slides toward their mass center—but not all the way! The agent might have to stop before hitting the mass center. When? This is where the q_i's come in. They ensure that the q_i-weighted mass center of the whole system stays put. Not to have that mass center wiggle around and produce exponentially small coefficients is a key reason why reversibility implies faster convergence. We flesh out this intuition in section 1.3.

The matrix approach. It is customary to model agreement systems by using products of stochastic matrices: $x(t+1) = P(t)x(t)$, where $x(t) = (x_1(t), \ldots, x_n(t))^T$ and $P(t)$ is a row-stochastic matrix whose entries $p_{ij}(t)$ are positive for all i, j with

$j \in N_i(t)$. Bidirectionality means that both $p_{ij}(t)$ and $p_{ji}(t)$ should be positive or zero, which is a form of *mutual confidence* [27]. Typically, one also requires a uniform lower bound on *each* nonzero entry of the matrices. We observe that condition (3) is not nearly as demanding: all we require is that if the agent i has at least one neighbor (besides itself), then the entries corresponding to the leftmost and rightmost neighbors $l(i)$ and $r(i)$ should be at most $1-\rho$. These conditions have a natural interpretation that we summarize as follows: For all t,

(5)
$$\begin{cases} \textit{Mutual confidence}\text{: No pair } p_{ij}(t), p_{ji}(t) \text{ has exactly one zero.} \\ \textit{No extreme influence}\text{: For any nonisolated agent } i, \max\{p_{il(i)}(t), p_{ir(i)}(t)\} \le 1-\rho. \end{cases}$$

Conditions (5) are weaker than the usual set of three constraints associated with the bidirectional case [18, 27], which, besides mutual confidence, include *self-confidence* (nonzero diagonal entries) and *nonvanishing confidence* (lower bound on all nonzero entries). Our model requires bounds on only two entries per matrix row. Previous work [2, 6, 27] highlighted the importance of self-confidence ($p_{ii}(t) > 0$) for the convergence of agreement systems. Our results refine this picture as follows: *To reach harmony in a group with bidirectional communication, individuals may be influenced extremely by nonextreme positions but must be influenced nonextremely by extreme positions ($m_{i,t}$ or $M_{i,t}$).* In the case of a two-agent system, this maxim coincides with the need for self-confidence; in general, the latter is not needed. We conclude this comment about the matrix representation of agreement systems by emphasizing that the total s-energy seeks to move the focus away from the matrices themselves and, instead, to reason about the agents' motion in phase space and their temporal communication network.

Random walks and ergodicity. At the risk of oversimplifying, one might say that to understand agreement systems is to understand backward products of stochastic matrices,

$$P(t)P(t-1)\cdots P(1)P(0),$$

as t grows to infinity. Forward products $P(0)P(1)\cdots P(t)$, for $t \to \infty$, are different, but much can be inferred about them from the backward variety. A forward product of stochastic matrices models a random walk in a temporal network: imagine walking randomly in a graph that may change at every step. Such products have been studied by computational complexity theorists, who call them *colored random walks* [10, 11]. This connection suggests that a complete theory of agreement systems would need to include, as a special case, a theory of discrete-time Markov chains. As we shall see, the total s-energy allows us to retrieve classical mixing bounds for random walks in undirected graphs.

A general principle behind the convergence of products of stochastic matrices is that, if all goes well, as t grows, the product will tend to a matrix of rank one or a (possibly permuted) block-diagonal matrix with blocks of rank one. Many analytical devices have been designed to keep track of this evolution, most of which fall into the category of *ergodicity coefficients* [38]. There is a simple geometric interpretation of this which is worth a short detour. From a stochastic matrix such as $P(0)$, construct a convex polytope by taking the convex hull, denoted $\texttt{conv}P(0)$, of the points formed by the rows of $P(0)$: here, each row forms the n coordinates of a point. When we multiply $P(1)$ by $P(0)$, each row of the product is a convex combination of the rows of $P(0)$, so the corresponding point lies inside the convex hull $\texttt{conv}P(0)$; therefore

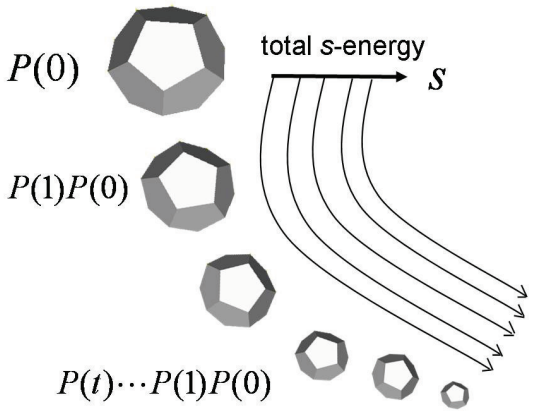

FIG. 3. *The total s-energy is a global instrument for tracking how fast the polytopes shrink.*

$\texttt{conv}\{P(1)P(0)\} \subseteq \texttt{conv}P(0)$. A coefficient of ergodicity is typically a measure of how quickly the nesting shrinks; i.e., it might keep track of the width, diameter, volume, or any other "shrinking" indicator as follows:

$$\texttt{conv}\{P(t)\cdots P(0)\} \subseteq \texttt{conv}\{P(t-1)\cdots P(0)\} \subseteq \cdots \subseteq \texttt{conv}\{P(1)P(0)\} \subseteq \texttt{conv}P(0).$$

For example, when all the matrices are identical (as in a Markov chain), the spectral gap gives us an ℓ_2-norm tracker of the contraction. What all coefficients of ergodicity share is time locality; i.e., they measure the contraction from one step to the next. The total s-energy, instead, is a *global* instrument (see Figure 3). It monitors the shrinking over all time steps in a global fashion: the parameter s plays the role of frequency in Fourier analysis and allows us to choose the correct "frequency" at which we want to monitor the shrinking process. This gives us Chernoff-like bounds on the distribution of the edge lengths.

1.2. The total s-energy. There is no obvious reason why the total s-energy, as defined in (1), should *ever* converge, so we treat it as a formal series for the time being. We prove that it converges for any real $s > 0$, and we bound its maximum value, $E_n(s)$, over all moves and n-node graph sequences. We may assume that all the agents start out in the unit interval (which, of course, implies that they remain there at all times). The justification is that the total s-energy obeys a power-law under scaling: $x \mapsto Cx$ implies that $E_n(s) \mapsto C^s E_n(s)$. We also assume throughout the remainder of this paper that ρ is smaller than a suitable constant. All the proofs of the results below are deferred to section 3.

THEOREM 1.1. *The maximal total s-energy of an n-agent bidirectional agreement system with unit initial diameter satisfies*

$$E_n(s) \le \begin{cases} \rho^{-O(n)} & \text{for } s = 1, \\ s^{1-n}\rho^{-n^2-O(1)} & \text{for } 0 < s < 1. \end{cases}$$

There is a lower bound of $O(\rho)^{-\lfloor n/2 \rfloor}$ on $E_n(1)$ and of $s^{1-n}\rho^{-\Omega(n)}$ on $E_n(s)$ for n large enough, any $s \le s_0$, and any fixed $s_0 < 1$.

The asymptotic notation hides the presence of absolute constant factors. For example, $\rho^{-O(n)}$, $O(\rho)^{-\lfloor n/2 \rfloor}$, and $\rho^{-\Omega(n)}$ mean, respectively, at most ρ^{-an}, at least

$(b\rho)^{-\lfloor n/2 \rfloor}$, and at least ρ^{-cn} for some suitable constants $a, b, c > 0$. Since no edge length exceeds 1, $E_n(s) \leq E_n(1)$ for $s \geq 1$, and so the theorem proves the convergence of the total s-energy for all $s > 0$.

When the temporal network always stays connected, it is useful to redefine the total s-energy as the sum of the sth powers of the diameters. Its maximum value, for unit initial diameter, is denoted by

$$E_n^D(s) = \sum_{t \geq 0} \Big(\operatorname{diam} \{x_1(t), \ldots, x_n(t)\} \Big)^s.$$

In dimension $d = 1$, the diameter is the length of the smallest enclosing interval. The following result is the sole breach of our pledge to avoid any connectivity assumption.

THEOREM 1.2. *The maximal diameter-based total s-energy of a connected n-agent reversible agreement system with unit initial diameter satisfies*

$$n^{-2} E_n(s) \leq E_n^D(s) \leq \frac{2n}{s} \Big(\frac{2n}{\rho} \Big)^{s/2+1}$$

for all $0 < s \leq 1$.

We proceed with general remarks about the function $E(s)$. All of the terms in the series are nonnegative, so we can assume they are rearranged in nonincreasing order. This allows us to express the total s-energy as a general Dirichlet series

$$E(s) = \sum_{k \geq 1} n_k e^{-\lambda_k s}, \tag{6}$$

where $\lambda_k = -\ln d_k$ and n_k is the number of edges of length d_k. Thus, $E(s)$ is the Laplace transform of a sum of scaled Dirac delta functions centered at $x = \lambda_k$. This implies that the total s-energy can be inverted and, hence, provides a lossless encoding of the edge lengths.

We show that $E(s)$ converges for any real $s > 0$. By the theory of Dirichlet series [15], it follows that $E(s)$ is uniformly convergent over any finite region $\mathcal{D}$ of the complex plane within $\Re(s) \geq r$ for any $r > 0$; furthermore, the series defines an analytic function over $\mathcal{D}$. It is immediate to determine the maximum s-energy of a two-agent system with unit initial diameter. For $\rho = 1/2 - 1/2e$, $E_2(s) = \sum_t (1-2\rho)^{st} = 1/(1-e^{-s})$; therefore, writing $s = x + iy$, it satisfies (see Figure 4)

$$|E_2(s)| = 1/\sqrt{1 - 2e^{-x} \cos y + e^{-2x}}.$$

The singularities are the simple poles $s = 2\pi i k$ for all k. The maximal total s-energy can be continued meromorphically over the whole complex plane. Note that this is obviously false for *nonmaximal* s-energies; for example, the function $\sum_k e^{-sk!}$ is a valid total s-energy, but its singularities form a dense subset of its line of convergence (the imaginary axis), and hence an impassable barrier for any analytic continuation into $\Re(s) < 0$.

1.3. Convergence. When bounding the convergence rate of agreement systems, we face the obvious difficulty that an adversary can always make the temporal network free of any edges joining distinct nodes for as long as it wants, and then, at some point far into the future, add all $\binom{n}{2}$ edges permanently to the temporal network in order to make all the agents cluster around the same point. How then can one hope to bound the convergence time since it can be arbitrarily large yet finite? The total s-energy is

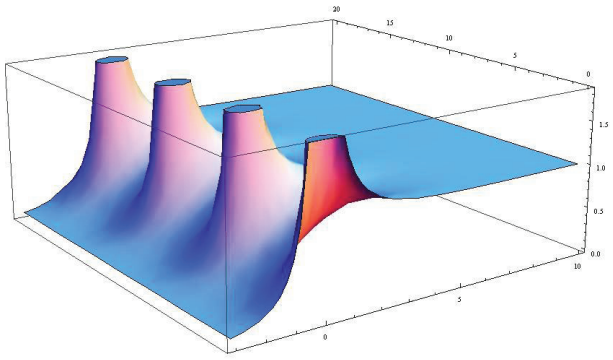

FIG. 4. *By analytic continuation, the maximum total s-energy of a two-agent system is a meromorphic function over the whole complex plane; the function is depicted in absolute value, with the real axis stretching to the right.*

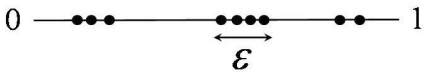

FIG. 5. *ε-convergence is reached when the agents fall within groups with disjoint enclosing intervals of length at most ε and no further interaction ever takes place between distinct groups.*

meant to deal with precisely that problem. Given $0 < \varepsilon < 1/2$, we say that a step t is *trivial* (where ε is understood) if all the edges of G_t have length at most ε. Recall that G_t always has n self-loops.[1] The *communication count* C_ε is defined as the total number of nontrivial steps. Intuitively, it is a way of ignoring microscopic motions. The system is said to *ε-converge* if the n agents can eventually be partitioned into subsets with disjoint enclosing intervals of length at most ε and, from that point on, no further interaction ever takes place between distinct subsets (Figure 5). *Consensus* refers to the case of a one-set partition. Visually, ε-convergence means that the system eventually freezes. We have this obvious relation between ε-convergence and triviality.

FACT 1.3. *An n-agent bidirectional agreement system $(n-1)\varepsilon$-converges by the time its last nontrivial step has elapsed. If the temporal network remains connected at all times, then the system $(n-1)\varepsilon$-converges to consensus within C_ε time.*

THEOREM 1.4. *The maximum communication count $C_\varepsilon(n)$ of any n-agent bidirectional agreement system with unit initial diameter satisfies*

$$O(\rho)^{-\lfloor n/2 \rfloor} \log \frac{1}{\varepsilon} \le C_\varepsilon(n) \le \min \left\{ \frac{1}{\varepsilon} \rho^{-O(n)}, \left(\log \frac{1}{\varepsilon} \right)^{n-1} \rho^{-n^2 - O(1)} \right\}.$$

If the initial diameter D is not 1, then we must replace ε by ε/D in the bounds for $C_\varepsilon(n)$. We easily check that the bound is essentially tight as long as ε is not superexponentially small. Indeed, for any constant $a > 0$, there exist two constants $b, c > 0$ such that, if $\varepsilon \ge \rho^{an}$, then (i) the communication count is at most $(1/\rho)^{bn}$,

[1] A self-loop always has zero length, but an edge (i, j) may be of zero length without being a self-loop.

and (ii) there exist an agreement system and a starting configuration for which the communication count exceeds $(1/\rho)^{cn}$. We put this more succinctly in the next corollary.

COROLLARY 1.5. *If $\varepsilon \geq \rho^{O(n)}$, then $C_\varepsilon(n) = \rho^{-\Theta(n)}$.*

THEOREM 1.6. *For any $0 < \varepsilon < \rho/n$, an n-agent reversible agreement system ε-converges to consensus in time $O(\frac{1}{\rho}\, n^2 \log \frac{1}{\varepsilon})$.*

Let $\delta = \max_{i,t} |N_i(t)|$ be the maximum degree of any node in the temporal network. The assignment $q_i = \delta$ is valid and gives $\rho = 1/\delta$. The theorem implies ε-convergence to consensus in $O(\delta n^2 \log \frac{1}{\varepsilon})$ time. A similar result holds if the degree of any given node does not change over time and each agent moves to the mass center of its neighbors as follows: $x_i(t+1) = (1/|N_i(t)|) \sum_{j \in N_i(t)} x_j(t)$. If we now consider the case of a time-invariant graph, we retrieve the usual polynomial mixing time bound for lazy random walks in undirected graphs.

The communication count is related to the total s-energy via the obvious inequality

$$C_\varepsilon \leq \varepsilon^{-s} E(s).$$

In view of this relation, the two upper bounds in Theorem 1.4 follow directly from those in Theorem 1.1: simply set $s = 1$ and $s = n/\ln \frac{1}{\varepsilon}$, respectively. Note that the second assignment can be assumed to satisfy $s < 1$, since it concerns only the case where $\frac{1}{\varepsilon}\rho^{-O(n)}$ is the bigger term in the right-hand side of the expression in Theorem 1.4. For reversible systems, we set $s = 1/\ln \frac{1}{\varepsilon}$ and observe that the number of steps witnessing a diameter in excess of ε is at most $\varepsilon^{-s} E_n^D(s) = O(\frac{1}{\rho}\, n^2 \log \frac{1}{\varepsilon})$. This bounds the time it takes for the diameter to dip below ε and stay there forever (since it cannot grow); hence Theorem 1.6 holds.

2. Applications. We highlight the utility of the total s-energy by looking at the following five examples: opinion dynamics (section 2.1), social epistemology (section 2.2), Kuramoto synchronization (section 2.3), bird flocking (section 2.4), and products of stochastic matrices (section 2.5).

2.1. Opinion dynamics. The Krause opinion dynamics model [16, 21] is a sociological framework for tracking opinion polarization in a population. In its d-dimensional version, the *bounded-confidence model*, as it is often called, sets a parameter $0 < r < 1$ and, at time 0, specifies the opinions of n agents as n points in the unit cube $[0,1]^d$. At time $t \geq 0$, each opinion x moves to the position given by averaging all the opinions that happen to fall within the Euclidean ball centered at x of radius r (or some other shape). Viewed as a multiagent agreement system, G_t consists of n nodes (the agents) with edges joining any two of them within a distance r of each other. The dynamics is specified by

$$x_i(t+1) = \frac{1}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t), \tag{7}$$

where $N_i(t)$ is the set of neighbors of node i in G_t, which as usual includes i itself. The system is known to converge [3, 21, 26, 27]. Theorem 1.4 allows us to bound how long it takes to reach equilibrium. Consider a Cartesian coordinate system. In view of (5, 7), we may set $p_{ij}(t) = 1/|N_i(t)|$ and $\rho = 1/n$ to make the opinion dynamics system along each coordinate axis conform to a one-dimensional multiagent agreement model (3). We can assume that the maximum diameter D along each axis is at most

rn at time 0 and hence remains so thereafter. Indeed, by convexity, if along any coordinate axis the n opinions have diameter greater than rn, then they can be split into two subsets with no mutual interaction now and forever. Set $\varepsilon = r/2$ and let t_ε be the smallest t such that G_t consists only of edges in R^d of length at most ε. During the first $dC_{\varepsilon/\sqrt{d}}(n) + 1$ steps, it must be the case that, at some time t, the graph G_t contains only edges of length at most ε. By Theorem 1.4, therefore

$$t_\varepsilon \leq d^{3/2}\frac{D}{\varepsilon}\, n^{O(n)} = 2d^{3/2}\, n^{1+O(n)} = n^{O(n)}. \tag{8}$$

Each connected component of G_{t_ε} is a complete graph. To see why, observe that if opinion x is adjacent to y in G_{t_ε} and the same is true of y and z, then x and z are at a distance of at most $2\varepsilon = r$, and hence are connected and therefore at a distance of at most ε at time t_ε. This "transitive closure" argument proves our claim. This implies that the opinions within any connected component end up at the same position at time $t_\varepsilon + 1$. Of course, when two opinions are joined together they can never be separated. The argument is now easy to complete. Either G_{t_ε} consists entirely of isolated nodes, in which case the system is frozen in place, or it consists of complete subgraphs that collapse into single points. The number of distinct opinions decreases by at least one, so this process can be repeated at most $n - 2$ times. By (8), this proves that Krause opinion dynamics converges in $n^{O(n)}$ time. We summarize our result.

THEOREM 2.1. *Any initial configuration of n opinions in the bounded-confidence Krause model with equal-weight averaging converges to a fixed configuration in $n^{O(n)}$ time.*

Martinez et al. [30] have established a polynomial bound for the one-dimensional case, $d = 1$. While extending their proof technique to higher dimension might be difficult, a polynomial bound could well hold for any constant d. We leave this as an interesting open problem.

2.2. Truth-seeking systems. In their pioneering work in computer-aided social epistemology, Hegselmann and Krause considered a variant of the bounded-confidence model that assumes a *cognitive division of labor* [17]. The idea is to take the previous model and fix one agent, *the truth*, while keeping the $n-1$ others mobile. A *truth seeker* is a mobile agent joined to the truth in every G_t. All the other mobile agents are *ignorant*, meaning that they never connect to the truth via an edge, although they might indirectly communicate with it via a path. Any two mobile agents are joined in G_t whenever their distance is less than r. (Using open balls simplifies the proofs a little.) Hegselmann and Krause [17] showed that, if all the mobile agents are truth seekers, they eventually reach consensus with the truth. Kurz and Rambau [22] proved that the presence of ignorant agents cannot prevent the truth seekers from converging toward the truth. The proof is quite technical and the authors leave open the higher-dimensional case. We generalize their results to any dimension and, as a bonus, bound the convergence rate.

THEOREM 2.2. *Any initial configuration of n opinions in R^d in the truth-seeking model converges, with all the truth seekers coalescing around the truth. If, in addition, we assume that the initial coordinates of each opinion as well as the radius r are encoded as $O(n)$-bit rationals, then, after $n^{O(n)}$ time, all the truth seekers lie within a ball of radius $2^{-n^{cn}}$ centered at the truth for any arbitrarily large constant $c > 0$. Ignorant agents either lie in that ball or are frozen in place forever. This holds in any fixed dimension.*

Proof. Along each coordinate axis, a truth-seeking system falls within the fixed-agent agreement model and, as we saw in section 1.1, can be simulated by a $(2n-1)$-agent one-dimensional bidirectional agreement system with at most twice the initial diameter. (Note that the $2n-1$ agents do not form a truth-seeking system because there are no edges connecting the group of n original agents to its reflection.) Convergence follows from Fact 1.3. As we observed in the previous section, restricting ourselves to the equal-weight bounded confidence model allows us to set $\rho = 1/(2n-1)$. (We could easily handle more general weights, but this complicates the notation without adding anything of substance to the argument.) Kurz and Rambau [22] observed that the convergence rate cannot be bounded as a function of n and ρ alone because it also depends on the initial conditions (hence the need to bound the encoding length of the initial coordinates).

Set $\varepsilon = 2^{-bn}$ for some large enough constant $b > 0$, and define t_ε as the smallest t such that G_t consists only of edges not longer than ε. By the same projection argument we used in (8) and the observation that the initial diameter is $2^{O(n)}$, $t_\varepsilon = n^{O(n)}$. The subgraph of G_{t_ε} induced by the mobile agents consists of disjoint complete subgraphs. Indeed, the transitive closure argument of the previous section shows that the distance between any two agents within the same connected component is at most $2\varepsilon = 2^{1-bn} < r$ (the inequality following from the $O(n)$-bit encoding of r), and hence at most ε. For similar reasons, the truth agent cannot join more than one of these complete subgraphs (referring here and below to the original system and *not* the duplicated version); therefore, all the subgraphs consist of ignorant agents, except for one of them, which contains all the truth seekers and to which the truth agent is joined. This *truth group* might contain some ignorant agents as well, i.e., mobile agents not connected to the truth. For that reason, the truth group, in which we include the truth, is a connected subgraph that might not be complete. At time $t_\varepsilon + 1$, the truth group has collapsed into either a single edge with the truth at one end or a collinear three-agent system consisting of the truth, a truth seeker, and an ignorant agent. (We refer to a single agent or truth seeker although it may be a collection of several of them collapsed into one.) All the other complete subgraphs collapse into *all-ignorant* single agents. By Theorem 2.1, there is a time

$$t_0 = t_\varepsilon + n^{O(n)} = n^{O(n)} \tag{9}$$

by which the all-ignorant agents will have converged into frozen positions unless they get to join with agents in the truth group at some point.

Case I. Assume that the all-ignorant agents do not join with any agent in the truth group at any time $t > t_\varepsilon$; the truth group then behaves like a one-dimensional fixed-agent system with two or three agents embedded in R^d. We assume the latter, the former case being similar, only easier. We saw in section 1.1 how such a system can be simulated by a one-dimensional five-agent bidirectional system of at most twice the diameter. Recall that agents may represent the collapse of several of them, so we must keep the setting $\rho = 1/(2n-1)$. The five-agent system remains connected at all times (since its diameter cannot grow); therefore, by Fact 1.3 and Theorem 1.4, it β-converges to consensus by (conservatively) time $t_0 + n^{O(1)}(\log \frac{1}{\beta})^4$. By (9), this implies that, for any fixed $c_0 > 0$, the agents of the truth group are within a distance of $2^{-n^{c_0 n}}$ of the truth after $n^{O(n)}$ time.

Case II. Assume now that an all-ignorant agent z joins with an agent y of the truth group at time t_1 but not earlier in $[t_\varepsilon, t_1)$. That means that the distance $\|y(t_1)z(t_1)\|_2$ dips below r for the first time after t_ε. We want to show that $t_1 \leq t_0 + n^{O(n)}$, so we might as well assume that $t_1 > t_0$. Recall that t_0 is an upper

bound on the time by which the all-ignorant agents would converge if they never interacted again with the truth group past t_ε. Let L be the line along which the truth group evolves, and let σ be its (nonempty) intersection with the open ball of radius r centered at $z(t_1) = z(t_0)$. Note that σ cannot be reduced to a single point. This implies that the shortest *nonzero* distance Δ between the truth and the two endpoints of σ is well defined. (By definition, if the truth sits at one endpoint, Δ is determined by the other one.) We claim that

$$\Delta \geq 2^{-n^{O(n)}}. \tag{10}$$

Here is why. It is elementary to express Δ as a feasible value of a variable in a system of m linear and quadratic polynomials over m variables, where m is a constant (depending on d). The coefficients of the polynomials can be chosen to be integers over $\ell = n^{O(n)}$ bits. (We postpone the explanation.) We need a standard root separation bound [46]. Given a system of m integer-coefficient polynomials in m variables with a finite set of complex solution points, any nonzero coordinate has modulus at least $2^{-\ell\gamma^{O(m)}}$, where $\gamma - 1$ is the maximum degree of any polynomial and ℓ is the number of bits needed to represent any coefficient. This implies our claimed lower bound of $2^{-n^{O(n)}}$ on Δ.

Why is $\ell = n^{O(n)}$? At any given time, consider the rationals describing the positions of the n agents and put them in a form with one common denominator. At time 0, each of the initial positions now requires $O(n^2)$ bits (instead of just $O(n)$ bits). A single time step produces new rationals whose common denominator is at most $n!$ times the previous one, while the numerators are sums of at most n previous numerators, each one multiplied by an integer at most $n!$. This means that, at time t, none of the numerators and denominators require more than $O(n^2 + tn \log n)$ bits. The system of equations expressing Δ can be formulated using integer coefficients with $O(n^2 + t_0 n \log n)$ bits; hence the bound $\ell = n^{O(n)}$. Next, we distinguish between two cases.

- *The truth is not an endpoint of* σ: In this case there is a closed segment of L centered at the truth that lies either entirely outside of σ or inside of it. By (10), the segment can be chosen to be of length at least $2^{-n^{O(n)}}$. Setting c_0 large enough, as we saw earlier, the agents of the truth group are within a distance of $2^{-n^{c_0 n}}$ of the truth after $n^{O(n)}$ time; therefore, $t_1 \leq t_0 + n^{O(n)}$, or else the diameter of the truth group becomes too small to accommodate Δ.
- *The truth is an endpoint of* σ: Quite clearly, β-convergence alone does not suffice to bound t_1, so we reason as follows. When the truth group has β-converged (for the previous value of β), the only way its mobile agents avoided falling within σ (in which case the previous bound on t_1 would hold) is if the truth group ended up separated from σ by the truth (lest one of the mobile agents lay in σ). By convexity, however, this property remains true from then on, and so z can never join y, which contradicts our assumption.

When agents y and z join in G_t at time $t = t_1$, their common edge is of length at least $r/3$ unless y or z has traveled a distance of at least $r/3$ between t_ε and t_1. In all cases, the system must expend 1-energy at least $r/3$ during that time interval. By Theorem 1.1, this can happen at most $n^{O(n)}(3/r) = n^{O(n)}$ times. We can repeat the previous argument safely each time, even though the bit lengths will increase. At the completion of this process, we are back to Case I. ∎

2.3. Kuramoto synchronization. The *Kuramoto model* is a general framework for nonlinear coupled oscillators, with a dazzling array of applications: circadian

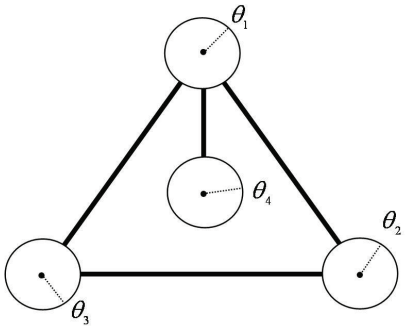

FIG. 6. *Four coupled oscillators connected by four edges.*

neurons, chirping crickets, microwave oscillators, yeast cell suspensions, pacemaker cells in the heart, etc. Winfree's pioneering work on the subject led Kuramoto to formulate the standard sync model for coupled oscillators [39, 45]. The system consists of n oscillators: the ith one has phase θ_i and natural frequency ω_i. In its original formulation, the model is a mean-field approximation that assumes all-pair coupling. A more realistic assumption is to use a time-varying network to model communications. Considerable work has been done on this problem; see [13, 20, 25, 29, 31, 35, 36, 44, 47] for a small sample. Further research introduced a time-1 discretization of the continuous model [29, 33, 37, 40]. Assuming that all oscillators share the same natural frequency, a fixed phase shift yields the dynamics

$$\theta_i(t+1) = \theta_i(t) + \frac{K\Delta T}{|N_i(t)|} \sum_{j \in N_i(t)} \sin(\theta_j(t) - \theta_i(t)),$$

where $|N_i(t)|$ is the degree of i in the communication graph G_t, which, as always, counts the self-loop at i (see Figure 6). As in [33], we also assume that all the agents' phases start in the same open half-circle. By shifting the origin, we express this condition as $\alpha - \pi/2 \le \theta_i(0) \le \pi/2$ for some arbitrarily small positive constant α. This implies that

$$\sin(\theta_j(0) - \theta_i(0)) = a_{ij}(\theta_j(0) - \theta_i(0))$$

for $\alpha/4 \le a_{ij} \le 1$. By (5), therefore, to make the dynamics conform to a bidirectional multiagent agreement system at time 0, it suffices to enforce the constraints

$$\frac{4n\rho}{\alpha} \le K\Delta T \le 1 - \rho\,.$$

Choosing $\rho = b\alpha/n$ for a small enough constant $b > 0$, we note that the constraints are roughly equivalent to $0 < K\Delta T < 1$. By convexity, the angles at time 1 remain within $[\alpha - \pi/2, \pi/2]$; therefore, our previous argument can be repeated to show that the synchronization dynamics fits within the bidirectional agreement model at all times. The result below follows from Corollary 1.5. We note that it is impossible to bound the actual time to convergence unless we make assumptions about the temporal network.

THEOREM 2.3. *Any Kuramoto synchronization system with n oscillators sharing the same natural frequency and initialized in an open half-circle ε-converges after $n^{O(n)}$ nontrivial steps for any $\varepsilon > n^{-cn}$ and any constant $c > 0$. This holds regardless of the temporal communication network.*

2.4. Bird flocking. Beginning with Reynolds's pioneering work in the mid-1980s, bird flocking has generated an abundant literature, with a sudden flurry of interest in the last few years. Mathematically, flocking appears more complex than the previous agreement systems because the averaging and the communications do not operate over precisely the same objects: it is the velocities that are averaged, but the positions (i.e., the integrals of the velocities) that determine the temporal network. Many models have been studied in the literature, but most of them are variants of the following [6, 12, 19, 43]: Given the initial conditions $z(0)$ and $z(1)$, for any $t > 0$,

$$\begin{cases} z(t) = z(t-1) + v(t), \\ v(t+1) = P(t)v(t). \end{cases} \tag{11}$$

The vectors $z(t), v(t)$ encode the positions and velocities of the n birds in R^3, and each coordinate of $z(t)$ and $v(t)$ is itself a three-dimensional vector. (These vectors are often expressed in R^{3n} via a tensor product; the notation here is easier as long as one remembers that the coordinates are themselves three-dimensional vectors.) The n-by-n stochastic matrix $P(t)$ has nonzero diagonal entries, and its other positive entries correspond to the edges of G_t; the communication graph G_t links any two birds within a fixed distance of each other. Intuitively, each bird averages out its own velocity with those of its neighbors in G_t: all of its neighbors weigh equally in the average except perhaps for itself; i.e., for fixed i, all nonzero $p_{ij}(t)$'s are equal, with the possible exception of $p_{ii}(t)$; all the entries in $P(t)$ are rationals over $O(\log n)$ bits.

It suffices to set $\rho = n^{-b}$, for a large enough constant $b > 0$, to make flocking conform to the bidirectional multiagent agreement model, with $v(t)$ encoding into a single vector the n points $(x_1(t), \ldots, x_n(t))$. By Corollary 1.5, the system ε-converges within $n^{O(n)}$ nontrivial steps for $\varepsilon \geq n^{-cn}$ and any constant $c > 0$. We showed in [7] that the sequence G_t always converges to a fixed graph G but that the number of steps to get there can be astronomical: it can be as high as a tower-of-twos of height on the order of $\log n$, which, amazingly, is tight.

THEOREM 2.4. *The velocities of n birds ε-converge after $n^{O(n)}$ nontrivial steps for any $\varepsilon > n^{-cn}$ and any constant $c > 0$. The number of steps prior to the convergence of the temporal network to a fixed graph is no higher than a tower-of-twos of height $O(\log n)$; this bound is optimal in the worst case.*

2.5. Products of stochastic matrices. Let $\mathcal{P}$ be the family of n-by-n stochastic matrices such that each $P \in \mathcal{P}$ satisfies the following three standard constraints: (i) *self-confidence* (nonzero diagonal entries); (ii) *mutual confidence* (no pair p_{ij}, p_{ji} with exactly one 0); and (iii) *nonvanishing confidence* (positive entries at least ρ). Lorenz [27] and Hendrickx and Blondel [18] independently proved the following counterintuitive result: In any finite product of matrices in $\mathcal{P}$, each nonzero entry is at least $\rho^{O(n^2)}$. What is surprising is that this lower bound is uniform, in that it is independent of the number of multiplicands in the product. We improve this lower bound to its optimal value as follows.

THEOREM 2.5. *Let $\mathcal{P}$ be the family of n-by-n real stochastic matrices such that any $P \in \mathcal{P}$ satisfies the following: each diagonal entry is nonzero; no pair p_{ij}, p_{ji} contains exactly one zero; and each positive entry is at least ρ. In any finite product of matrices in $\mathcal{P}$, each nonzero entry is at least ρ^{n-1}. The bound is optimal.*

3. The proofs. It remains for us to prove Theorem 1.1 (upper bound for $s = 1$ in section 3.1, upper bound for $s < 1$ in section 3.2, and lower bounds in section 3.4), Theorem 2.5 (section 3.2), Theorem 1.2 (section 3.3), and the lower bound of Theorem 1.4 (section 3.4).

3.1. The general case: $s = 1$. We prove the upper bound of Theorem 1.1 for $s = 1$. We show that $E_n(1) \leq \rho^{-O(n)}$ by bounding the kinetic s-energy.

Wingshift systems. We introduce a *wingshift system*, which provides a simpler framework for the proof (see Figure 7). Since we focus on a single transition at a time, we write a_i, b_i instead of $x_i(t), x_i(t+1)$ for notational convenience, and we relabel the agents so that $0 \leq a_1 \leq \cdots \leq a_n \leq 1$. Given $a_1, \ldots, a_n$, the agents move to their next positions $b_1, \ldots, b_n$ and then repeat this process endlessly in the manner described below. Let $\ell(i)$ and $r(i)$ be indices satisfying the following inequalities:

RULE 1: $1 \leq \ell(i) \leq i \leq r(i) \leq n$ and $(\ell \circ r)(i) \leq i \leq (r \circ \ell)(i)$.

RULE 2: $a_{\ell(i)} + \delta_i \leq b_i \leq a_{r(i)} - \delta_i$, where $\delta_i = \rho(a_{r(i)} - a_{\ell(i)})$.

Each agent i picks an *associate* to its left (perhaps itself) and one to its right, $\ell(i)$ and $r(i)$, respectively. It then shifts anywhere in the interval $[a_{\ell(i)}, a_{r(i)}]$, though keeping away from the endpoints by a small distance δ_i. This process is repeated forever, with each agent given a chance to change associates at every step. Any multiagent agreement system with parameter ρ can be modeled as a wingshift system: each agent picks its leftmost and rightmost neighbors as associates; note that the wingshift graph is sparser but now dependent on the embedding. Bidirectionality ensures Rule 1: it says that the interval $[\ell(i), r(i)]$ should contain i as well as all agents j pointing to i.[2] By analogy, we define the total 1-energy of the wingshift system as $V = \sum_{t \geq 0} V_t$, where (with a_i denoting $x_i(t)$) $V_t = \sum_{i=1}^{n} (a_{r(i)} - a_{\ell(i)})$. The desired upper bound $E_n(1) = \rho^{-O(n)}$ follows trivially from this bound on V as follows.

THEOREM 3.1. *The maximal total* 1*-energy of an* n*-agent wingshift system with unit initial diameter and parameter* ρ *is at most* $\rho^{-O(n)}$.

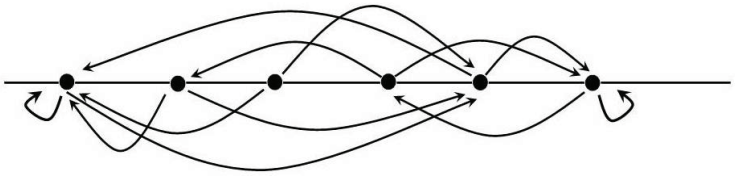

FIG. 7. *A six-node wingshift system.*

As usual, we assume that ρ is smaller than a suitable constant. We need some notation to describe rightward paths in the wingshift system: $r(i, 0) = i$ and $r(i, k) = r(r(i, k-1))$ for $k > 0$. We define the distance between an agent and its right associate, $\Delta_i = a_{r(i)} - a_i$. See Figure 8. When traversing a rightward path $i = r(i, 0), r(i, 1), \ldots, r(i, k)$, etc., a sudden drop in $\Delta_{r(i,k)}$ is of particular interest, so we introduce $r_i = r(i, k_i)$, where

$$k_i = \min \left\{ k \geq 0 \,|\, \Delta_{r(i,k)} \geq \frac{2}{\rho} \Delta_{r(i,k+1)} \right\}.$$

[2] This is necessary for convergence. Consider three agents $a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = 1$, with $\ell(1) = r(1) = \ell(2) = 1$ and $r(2) = \ell(3) = r(3) = 3$. Agents 1 and 3 are stuck in place while agent 2 can move about freely forever.

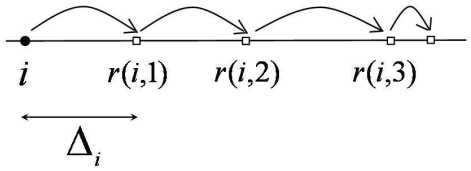

FIG. 8. *The stopper of agent i is $r(i,3)$, $r_i = r(i,2)$, and $k_i = 2$.*

The agent $r(r_i)$ is called the *stopper* of i. Our interest in stoppers is that the nontrivial ones always move left, thus pointing to some obligatory motion in the system at the time step under consideration.

LEMMA 3.2. *The stopper u of any agent i satisfies $a_u - b_u \geq (\frac{\rho}{2})^{k_i+1}\Delta_i$.*

Proof. By Rule 1, $a_{r(u)} - a_{\ell(u)} \geq \Delta_{r_i} + \Delta_u$. Since $\Delta_u \leq \frac{\rho}{2}\,\Delta_{r_i}$, it follows by Rule 2 that

$$\begin{aligned} b_u &\leq a_{r(u)} - \rho(a_{r(u)} - a_{\ell(u)}) \leq a_u + \Delta_u - \rho(\Delta_{r_i} + \Delta_u) \\ &\leq a_u + ((1-\rho)\rho/2 - \rho)\Delta_{r_i} \leq a_u - (1+\rho)(\rho/2)^{k_i+1}\Delta_i\,. \end{aligned}$$

The last inequality follows from the fact that $\Delta_{r_i} \geq (\rho/2)^{k_i}\Delta_i$; it is not strict because both sides are equal if $k_i = 0$. ∎

We bound V by tallying the kinetic 1-energy of the system, $K = \sum_{t\geq 0} K_t$, where $K_t = \sum_{i=1}^{n} |a_i - b_i|$. Recall that a_i and b_i are the positions of agent i at times t and $t+1$, respectively.

LEMMA 3.3. *If K is finite, then $V \leq n2^n\rho^{1-n}K$.*

Proof. Obviously,

$$V = \sum_{t\geq 0}\sum_{i=1}^{n}(a_{r(i)} - a_{\ell(i)}) \leq \sum_{t,i}\Big\{(a_i - a_{\ell(i)}) + (a_{r(i)} - a_i)\Big\} \leq \sum_{t,i}\Delta_i + \sum_{t,i}\Delta_i'\,,$$

where $\Delta_i' = a_i - a_{\ell(i)}$. By Lemma 3.2,

$$\sum_{t\geq 0}\sum_{i=1}^{n}\Delta_i \leq \sum_{t\geq 0}\sum_{i=1}^{n}\Big(\frac{2}{\rho}\Big)^{k_i+1}(a_{r(r_i)} - b_{r(r_i)}) \leq n\Big(\frac{2}{\rho}\Big)^{n-1}\sum_{t\geq 0}\sum_{i=1}^{n}|a_i - b_i|\,.$$

A mirror-image argument yields the same upper bound on $\sum_{t,i}\Delta_i'$. ∎

The idea behind the proof. By symmetry, we can assume that at least half of the contribution to K is provided by rightward motions, i.e., $\frac{1}{2}K \leq \sum_{t,i}\{\, b_i - a_i \,|\, b_i > a_i \,\}$. Thus we can conveniently ignore all leftward travel for accounting purposes. We use an "amortization" technique that involves assigning a *credit account* to each agent. Whenever the agent moves right it is required to pay for its travel cost by decreasing its account by an amount equal to the distance it travels. Credits are injected into the system only at time 0; if all travel is paid for and no account is overdrawn, then clearly the initial injection is an upper bound on $\frac{1}{2}K$. The benefit of this approach is that accounts can borrow from one another, thus creating an "economy" of credits. The proof takes the form of an algorithm that drives the trading in a manner that keeps all accounts solvent; in other words, it is an *algorithmic proof* [8].

Agent i cannot, in a single step, move to the right by a distance greater than Δ_i. Lemma 3.2 suggests a paying mechanism by which we charge its stopper $u = r(r_i)$

that travel cost; in other words, the leftward travel of u would pay for the rightward travel of the agents that claim u as a stopper. If u moves only to the left, then its own travel distance is bounded by 1, and the charging scheme is essentially sound. But what if u zigzags left and right? The premise of charging u for the cost of i is that we know how to bound the cost of u. But, if u moves in both directions, we cannot bound its cost a priori (whereas we can if it only travels left). The solution is to look at u's own stopper u' and charge it. This may, in turn, force u' to charge u'', etc. The "buck passing" evolves from left to right, so it must eventually stop. This picture suggests that agents should hold more credits the further to the right they are; indeed, our credit invariant will relate an agent's account to its rank.

The algorithmic proof. At time t, the agents are ordered as $0 \le a_1 < \cdots < a_n \le 1$. By using standard perturbation techniques, we can assume strict inequalities among all agent positions at all times. We maintain the following credit invariant: At the beginning of each time step, every agent i holds $a_i\alpha^i$ credits in its account, for some fixed parameter α, where again a_i is shorthand for $x_i(t)$. By way of illustration, consider the trivial case of two agents, one at 0 and the other at 1, meeting at $1-\rho$ at the next step (we use ties for convenience). The system holds α^2 at time 0 and $(1-\rho)(\alpha+\alpha^2)$ at time 1. The difference exceeds the travel cost of $1-\rho$ if α is sufficiently larger than $1/\rho$.

Our algorithmic proof involves setting up a simple data structure, a linked list, and moving credits around according to specific rules. Let u be a stopper such that $b_u < a_u$. Consider the lowest-ranked agent h that claims u as its stopper. We build a doubly linked list L_u consisting of $u-h+1$ nodes, each one corresponding to an agent: h is at the head and u at the tail; scanning L_u takes us through the agents $h, h+1, \ldots, u$. The nodes scanned after v are called the *antecedents* of v. The *rank* $s(v)$ is h plus the number of steps it takes to get from h to v in L_u. Ranks are implied by the list, so that inserting a node automatically adds one to the ranks of its antecedents. Initially, the rank of the node v corresponding to agent k is just k, and its *position*, denoted by $a(v)$, is a_k. The node following (resp., preceding) v in L_u, if it exists, is denoted by $\texttt{next}(v)$ (resp., $\texttt{prev}(v)$). We identify the node m with the highest-ranked agent such that $a_m < b_u$. Let $\beta = \rho/2$ and $\alpha = 6/\rho^2$ (see Figure 10).

Step [1]. Since $b_u < a_u$, node u is the stopper of at least one node strictly to its left, so $|L_u| > 1$ and m is well defined. By assigning $a(w) = b_u$, in effect we move the stopper u to its new position b_u, right after agent m in L_u. Shifting accounts one step backward gets w to inherit the account of $m+1$ and the new tail to receive the credits formerly at u. If $m = u-1$, step [1] ends with the list in the same state as before except for $a(\texttt{tail})$; otherwise, the $u-m-1$ antecedents of w see their ranks automatically incremented by one, and, among them, the node for any agent k acquires the credit account of $k+1$. (The alternative of keeping the list intact and shifting positions $a(v)$ to the right works but breaks the immutable correspondence between nodes and agents.) To summarize, at the end of step [1], any node v in L_u ends up with $a(v)\alpha^{s(v)}$ credits if v comes before w in the list and $a(\texttt{next}(v))\alpha^{s(v)}$ otherwise (see Figure 9).[3]

Step [2]. We prove that all the credit allocations are feasible. Suppose that v is either w or an antecedent of w. Agent v has $a(\texttt{next}(v))\alpha^{s(v)}$ credits. It receives $\beta(a_u - a(\texttt{next}(v)))\alpha^{s(\texttt{next}(v))}$ credits from $\texttt{next}(v)$ (which, by our notational convention, is zero if v is the new tail); it also gives away $\beta(a_u - a(v))\alpha^{s(v)}$ credits and keeps

[3]By abuse of notation, $a(\texttt{next}(\texttt{v}))$ denotes a_u if v is the new tail.

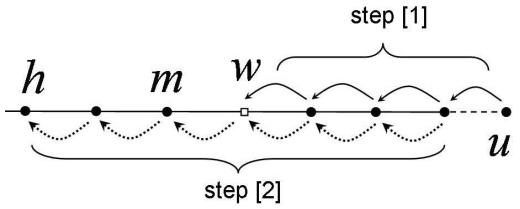

FIG. 9. *Credits are transferred via plain arrows in step* [1] *and dashed arrows in step* [2]. *Position* $a(v) = a_h, a_m, b_u, a_u$ *for* $v = h, m, w, u$.

- For each stopper u from right to left, if $b_u < a_u$, do:
  - [1] Insert a new node w into L_u right after m: $\texttt{next}(w) \leftarrow \texttt{next}(m)$ and $\texttt{next}(m) \leftarrow w$. Set $a(w) = b_u$. For each antecedent v of w, transfer the account of v to $\texttt{prev}(v)$. Delete node u from L_u.
  - [2] For each node v from the new tail to $\texttt{next}(h)$:
    - transfer $\beta(a_u - a(v))\alpha^{s(v)}$ credits from v to $\texttt{prev}(v)$;
    - keep $(\beta a(v) + (1-\beta)a_u)\alpha^{s(v)}$ credits in the account of v.
  - [3] Move from a_i to b_i any agent i claiming u as its stopper, provided that $b_i > a_i$. Enforce all credit invariants.
- Move from a_i to b_i any nonstopper i such that $b_i < a_i$. Enforce all credit invariants.

FIG. 10. *The algorithmic proof.*

$(\beta a(v) + (1-\beta)a_u)\alpha^{s(v)}$ of them, for a total need of $a_u\alpha^{s(v)}$. Since $s(\texttt{next}(v)) = s(v) + 1$, the transaction balances out if

$$a(\texttt{next}(v))\alpha^{s(v)} + \beta(a_u - a(\texttt{next}(v)))\alpha^{s(v)+1} \geq a_u\alpha^{s(v)},$$

which holds because $\alpha\beta \geq 1$. Suppose now that v comes before w in the list. The only difference is that v now starts out with an account worth $a(v)\alpha^{s(v)}$ credits. The balance condition becomes

$$a(v)\alpha^{s(v)} + \beta(a_u - a(\texttt{next}(v)))\alpha^{s(v)+1} \geq a_u\alpha^{s(v)},$$

which is equivalent to

$$a_u - a(v) \leq \alpha\beta(a_u - a(\texttt{next}(v))). \tag{12}$$

To see why (12) holds, we turn to the wingshift condition—as, at some point, we must. Among the agents of L_u claiming u as a stopper and v as an antecedent, let z be the last one. (Note that z may not be equal to v, but because of h it is sure to exist.) Extending the notation in the obvious way, by Lemma 3.2, $\Delta_z \leq (\frac{2}{\rho})^{k_z+1}(a_u - b_u)$. If $k_z = 0$, then

$$\frac{a_u - a(v)}{a_u - a(\texttt{next}(v))} \leq \frac{a_u - a_z}{a_u - b_u} = \frac{\Delta_z}{a_u - b_u} \leq \frac{2}{\rho} < \alpha\beta,$$

which proves (12). If $k_z > 0$, then, by the maximality of z,

$$a(\texttt{next}(v)) \le a_{r(z)} \le a_{r(r(z))} \le a_u,$$

and condition (12) follows from

$$\frac{a_u - a(v)}{a_u - a(\texttt{next}(v))} \le \frac{a_u - a_z}{a_u - a_{r(z)}} = \frac{\Delta_z + \Delta_{r(z)} + a_u - a_{r(r(z))}}{\Delta_{r(z)} + a_u - a_{r(r(z))}} \le \frac{\Delta_z + \Delta_{r(z)}}{\Delta_{r(z)}} \le 1 + \frac{2}{\rho}.$$

Step [3]. Having shown that all the accounts can afford the amounts specified in the second bullet of step [2], we now explain how they can pay for all rightward travel. We use an accounting trick, which is to move agents *without crossing*: all agents move continuously, one at a time. Should agent i bump into agent j, the latter completes the former's journey while i stops; the process is repeated at each collision. The main advantage of this scheme is that agents now keep their ranks at all times (of course they must also swap identities, thus becoming *virtual* agents). Unlike before, a rightward move $a_i \to b_i$ will now entail the motion of all the virtual agents in the interval $[a_i, b_i]$ and not only those claiming u as their stopper. By step [2], each node v is supplied with $(\beta a(v) + (1-\beta)a_u)\alpha^{s(v)}$ credits, which is at least

$$a(v)\alpha^{s(v)} + (1-\rho)(a_u - a(v))\alpha^{s(v)} + a_u - a(v),$$

because $s(v) \ge 1$. The three-part sum shows explicitly why virtual agent v, whose rank is now fixed, can move right by a distance of at least $(1-\rho)(a_u - a(v))$ while both maintaining its credit invariant and paying (comfortably) for the travel cost. Virtual agent v never needs to move further right than that. Why is that? The motion might be generated by v itself (if u is its stopper) or by the "push" from an agent on its left. Either way, at any instant during the continuous motion of virtual v, there is a causing agent i (perhaps v itself) whose corresponding interval $[a_i, b_i]$ contains the position of v at that instant. Our claim follows then from Rule 2. Indeed,

$$b_i \le a_{r(i)} - \rho(a_{r(i)} - a_{\ell(i)}) \le \rho a_{\ell(i)} + (1-\rho)a_{r(i)} \le \rho a(v) + (1-\rho)a_u \le a(v) + (1-\rho)(a_u - a(v)).$$

Returning all agents to their nonvirtual status, we observe that processing stopper u moves to the right only the agents that claim it as a stopper. Treating stoppers u in descending order from right to left means that none of the agents with u as their stopper have yet been moved (either to the left as stoppers or to the right) by the time we handle u. The last step in the algorithm can only release credits—think of virtual agents to see why—and so, maintaining the corresponding invariants is immediate. This allows us to bound the kinetic 1-energy by[4]

$$\frac{1}{2}K \le \sum_{i=1}^{n} x_i(0)\alpha^i \le 2\alpha^n = \rho^{-O(n)}.$$

Theorem 3.1 now follows from Lemma 3.3, which completes the proof of the upper bound of Theorem 1.1 for $s = 1$. ∎

[4]A more sophisticated argument allows us to lower α to $O(1/\rho)$ and thus reduce the constant in the $O(n)$ exponent to 1; this sort of finetuning is not needed for the purposes of this paper.

3.2. The general case: `s < 1`. We prove the upper bound of Theorem 1.1 for $0 < s < 1$. We show that the total s-energy satisfies the recurrence: $E_1(s) = 0$ and, for $n \geq 2$,

$$E_n(s) \leq 2nE_{n-1}(s) + (1 - (\rho/2)^n)^s E_n(s) + n^3. \tag{13}$$

We prove (13) "algorithmically" by describing a procedure to track the propagation of information across the temporal network and monitor its effect on the geometry of the system. All agents are initially *dry*, except for agent 1, which is *wet*. Every time a wet agent communicates with a dry one, the latter becomes wet. Once wet, an agent always remains so. Through the communication provided by the temporal network, water propagates from agent to agent. Bidirectionality ensures that, when the water ceases to spread to dry nodes, the interval spanned by the wet agents will be expected to have shrunken a little; in other words, communication acts as a spring that pulls recipients together (see Figure 11).

[1] Initially, all agents are dry except for agent 1. Set $W(0) = \{x_1(0)\}$.

[2] For $t = 0, 1, \ldots, \infty$:

- [2.1] Declare wet any agent adjacent to a wet agent in G_t.
- [2.2] $W\ (t) \leftarrow W(t) \cup \{$ positions at time t of dry agents just turned wet $\}$.
- [2.3] Move each agent i from $x_i(t)$ to $x_i(t+1)$. [`If no newly wet agent, then all motion within` $W(t) = W(t\ )$ `occurs in isolation from the` $n - |W(t)|$ `other agents.`]
- [2.4] $W(t+1) \leftarrow \{$ positions at time $t+1$ of agents corresponding to $W\ (t)\ \}$.

FIG. 11. *The flow algorithm.*

The set $W(t)$ tracks the positions of the wet agents at time t. The auxiliary set $W^*(t)$ includes the positions at time t of the agents wet at time $t+1$; it differs from $W(t+1)$ only in that the latter gives the positions at time $t+1$. Let $\|W(t)\|$ denote the length of the smallest interval enclosing $W(t)$, and let $\{t_k\}_{k\geq 1}$ be the times $t \geq 0$, in chronological order, at which $|W^*(t)| > |W(t)|$ (i.e., at least one dry agent turns wet at time t).[5] Recall that ρ is smaller than a suitable constant. We show that

$$\|W(t_k)\| \leq 1 - \left(\frac{\rho}{2}\right)^k. \tag{14}$$

The smallest interval $[a, b]$ defining $\|W(t_k)\|$ is in $[0, 1]$. By symmetry, we can always assume that $a + b \geq 1$. Because $\|W(t_1)\| = 0$, we can also safely assume by induction that (14) holds up to t_k; hence $a \geq \frac{1}{2}(\rho/2)^k$. Since $\|W(t)\|$ can increase only when at least one dry agent becomes wet, i.e., at times of the form $t = t_l$, we can prove (14) for t_{k+1} by showing that $\|W(t_k + 1)\| \leq 1 - (\rho/2)^{k+1}$. This easily follows from $[0, a\rho) \cap W(t_k + 1) = \emptyset$, so it suffices to prove the latter, which we do by contradiction (see Figure 12). Consider an agent i contributing to $W(t_k + 1)$ with $x_i(t_k + 1) < a\rho$. Agent i is wet at time $t_k + 1$, so at least one agent in $N_i(t_k)$ was wet at time t_k (possibly i itself). This implies that $M_{i,t_k} \geq a$ and, by (3),

$$x_i(t_k + 1) \geq (1 - \rho)m_{i,t_k} + \rho M_{i,t_k} \geq a\rho,$$

which is impossible and proves (14).

[5] Both $W(t)$ and $W\ (t)$ are understood as multisets. Note that t_k might not exist.

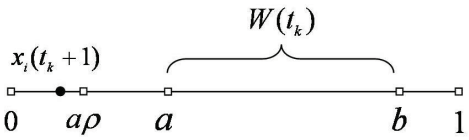

FIG. 12. *Bounding the interval spanned by wet agents.*

The set $W(t_k)$ can only gain agents, as k grows, but the set may stop growing before it absorbs all of them. When t is not of the form t_k, the agents of $W(t)$ interact only among themselves, so the total s-energy expended during steps $t_{k-1}+1, \ldots, t_k-1$ is bounded by $E_{|W(t_k)|}(s) + E_{n-|W(t_k)|}(s)$. At time $t = t_k$, the extra energy involved is

$$\sum_{(i,j)\in G_t} |x_i(t) - x_j(t)|^s \le \binom{n}{2}.$$

Using obvious monotonicity properties, it follows that, up to the highest value of t_k, the s-energy is bounded by

$$\sum_{l=1}^{n-1} \left\{ E_l(s) + E_{n-l}(s) + \binom{n}{2} \right\} < 2nE_{n-1}(s) + n^3.$$

This includes the case where no t_k exists. When it does and reaches its highest value t, if $|W(t+1)| < n$, then all the energy has been accounted for above. Otherwise, we must add the energy expended by the n agents past t. By (14), however, at time $t+1$, the n agents fit within an interval of length $1 - (\rho/2)^n$. By the scaling (power) law of the total s-energy, all we need to do is add $(1 - (\rho/2)^n)^s E_n(s)$ to the sum; hence (13) holds.

The case $n = 2$ is worthy of attention because it is easy to solve exactly. In the worst case, the two agents start at 0 and 1 and move toward each other by the minimum allowed distance of ρ. This gives us the equation $E_2(s) = (1-2\rho)^s E_2(s)+1$; hence, by (17) ahead,

$$E_2(s) = \frac{1}{1-(1-2\rho)^s} \le \frac{1}{2s\rho}. \tag{15}$$

We now consider the case $n > 2$. By (17) and (13),

$$E_n(s) \le \frac{2nE_{n-1}(s) + n^3}{s(\rho/2)^n}.$$

By (15) and the monotonicity of $E_n(s)$, we verify that the numerator is less than $3n^3 E_{n-1}(s)$; therefore, for $n > 2$, by (15),

$$E_n(s) < \frac{3n^3 E_{n-1}(s)}{s(\rho/2)^n} \le s^{1-n}\rho^{-n^2-O(1)}.$$

This proves the upper bound of Theorem 1.1 for $s < 1$. ∎

Proof of Theorem 2.5. Recall that $\mathcal{P}$ is the family of n-by-n stochastic matrices such that any $P \in \mathcal{P}$ satisfies the following: each diagonal entry is nonzero; no pair

p_{ij}, p_{ji} contains exactly one zero; and each positive entry is at least ρ. By (5), the entry (i,j) of a product of t such matrices can be viewed as the position of agent i after t iterations of a bidirectional system with agreement parameter ρ, initialized with all the agents at 0, except for j positioned at $x_j(0) = 1$. Referring back to the algorithm, we designate agent j as the one initially wet, with all the others dry. Let $m(t)$ be the minimum value in $W(t)$. At every time t_k when $W(t)$ grows in size, the minimum $m(t)$ cannot approach 0 closer than $\rho m(t)$. Since $|\{t_k\}| < n$, agent i either stays dry forever and does not leave 0 or joins $W(t)$ and cannot be smaller than $\min_t m(t)$, which is at least ρ^{n-1}. The lower bound proof suggests a trivial construction that achieves the very same bound and therefore proves its optimality. This completes the proof of Theorem 2.5. □

3.3. The reversible case. Our proof of Theorem 1.2 is based on a standard use of the Dirichlet form and classical spectral gap arguments [9, 23, 32]. Let $\pi_i = q_i / \sum_j q_j$. We easily verify that $\pi = (\pi_1, \dots, \pi_n)$ is the (time-invariant) stationary distribution of the stochastic matrix $P = P(t)$ specified by (4):

$$p_{ij} = \begin{cases} 1 - (|N_i| - 1)/q_i & \text{if } i = j, \\ 1/q_i & \text{if } i \neq j \in N_i, \\ 0 & \text{else.} \end{cases}$$

The argument focuses on a fixed step, so we may drop t to simplify the notation. Let $x = (x_1, \dots, x_n)$ and, for $u, v \in \mathsf{R}^n$, let $\langle u, v \rangle_\pi = \sum \pi_i u_i^T v_i$. The dynamics is invariant under translation, so we may move the origin to ensure that $\langle x, \mathbf{1} \rangle_\pi = 0$. Because π is the stationary distribution, this property is time-invariant; in particular, $\langle Px, \mathbf{1} \rangle_\pi = 0$. Because P is reversible, we can decompose $x = \sum_i a_i v_i$ in an eigenbasis $\{v_i\}$ for P orthonormal with respect to $\langle \cdot \rangle_\pi$; all the eigenvalues are real. Any positive p_{ij} is at least $1/q_i \geq \rho$. Let $1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq 2\rho - 1$ be the eigenvalues of P, with the labeling matching the v_i's. Why the inequalities? Briefly, the gap is strict between the two largest eigenvalues because the graph is connected; the smallest eigenvalue is separated from -1 by at least 2ρ because $(P - \rho I)/(1 - \rho)$ is itself a reversible Markov chain (with the same eigenvectors), and hence with real-valued spectrum in $[-1, 1]$. By the Perron–Frobenius theorem, if $\mu = \max\{\lambda_2^2, \lambda_n^2\}$ then, by reversibility, $\pi_i p_{ij} = \pi_j p_{ji}$, and

(16)
$$\begin{aligned} \langle x, x \rangle_\pi - \langle Px, Px \rangle_\pi = \langle x, (I - P^2)x \rangle_\pi &= \sum_{i,j} a_i a_j \langle v_i, (I - P^2) v_j \rangle_\pi = \sum_i a_i^2 (1 - \lambda_i^2) \\ &\geq (1 - \mu) \sum_i a_i^2 = (1 - \mu) \sum_{i,j} a_i a_j \langle v_i, v_j \rangle_\pi = (1 - \mu) \langle x, x \rangle_\pi. \end{aligned}$$

Because P is reversible and any nonzero $\pi_i p_{ij}$ is at least ρ/n, it holds that, for any vector z,

$$\langle z, (I - P)z \rangle_\pi = \frac{1}{2} \sum_{i,j} \pi_i p_{ij} (z_i - z_j)^2 \geq \frac{\rho}{2n} \sum_{(i,j) \in G_t} (z_i - z_j)^2.$$

Set $z = v_2$. By orthonormality, $\langle z, z \rangle_\pi = 1$ and $\langle z, \mathbf{1} \rangle_\pi = 0$; therefore, z must contain a coordinate z_a such that $|z_a| \geq 1$ and another one, z_b, of opposite sign. Since G_t is

connected, there is a simple path L connecting nodes a and b. By the Cauchy–Schwarz inequality,

$$1-\lambda_2 = \langle z,(I-P)z\rangle_\pi \geq \frac{\rho}{2n}\sum_{(i,j)\in L}(z_i-z_j)^2 \geq \frac{\rho}{2n^2}\left(\sum_{(i,j)\in L}|z_i-z_j|\right)^2$$
$$\geq \left(\frac{\rho}{2n^2}\right)(z_a-z_b)^2 = \frac{\rho}{2n^2}\,.$$

Since $\lambda_n+1\geq 2\rho$, it then follows that

$$\mu \leq \left(1-\frac{\rho}{2n^2}\right)^2 \leq 1-\frac{\rho}{2n^2}\,,$$

and, by (16),

$$\langle Px,Px\rangle_\pi \leq \mu\langle x,x\rangle_\pi \leq \left(1-\frac{\rho}{2n^2}\right)\langle x,x\rangle_\pi.$$

Let $E_n^D(L,s)$ be the maximum value of the (diameter-based) total s-energy of an n-agent reversible agreement system such that $\langle x,x\rangle_\pi = L$ at time 0. Since G_t is connected, $q_i\geq 2$; therefore the diameter is at most

$$2\max_i|x_i| \leq 2\sqrt{L/\min_i \pi_i} \leq \sqrt{2Ln/\rho};$$

therefore,

$$E_n^D(L,s) \leq E_n^D((1-\rho/2n^2)L,s) + (2Ln/\rho)^{s/2}.$$

The total s-energy obeys the scaling law $E_n^D(\alpha L,s)=\alpha^{s/2}E_n^D(L,s)$. The definition of $E_n^D(s)$ assumes unit initial diameter, which implies that $\langle x,x\rangle_\pi\leq 1$; hence $E_n^D(s)\leq E_n^D(1,s)$ and

$$E_n^D(s) \leq \frac{(2n/\rho)^{s/2}}{1-(1-\rho/2n^2)^{s/2}} \leq \frac{2n}{s}\left(\frac{2n}{\rho}\right)^{s/2+1},$$

which proves Theorem 1.2. This follows immediately from an inequality we use repeatedly. For any $0\leq a,b\leq 1$,

$$(1-a)^b \leq 1-ab. \qquad (17)$$ □

3.4. The lower bounds. We prove the lower bounds in Theorems 1.1 and 1.4.

The case $s<1$. We describe an algorithm $\mathcal{A}_n(a,b)$ that moves n agents initially within $[a,b]$ toward a single point $a+(b-a)y(n)$ while producing a total s-energy equal to $(b-a)^sE(n,s)$. Clearly, $E(1,s)=0$, so assume $n>1$. We specify $\mathcal{A}_n(0,1)$ as follows. Place $n-1$ agents at position 0 and one at position 1. The graph G_0 consists of a single edge between agent 1 at position 1 and agent 2 at position 0. At time 0, agent 2 moves to position ρ while agent 1 shifts to $1-\rho$. The $n-2$ other agents stay put. Next, apply $\mathcal{A}_{n-1}(0,\rho)$ to the set of all agents except 1. By induction, we can assume that this brings them to position $\rho y(n-1)$. Finally, apply $\mathcal{A}_n(\rho y(n-1),1-\rho)$ to all the agents. The operations of $\mathcal{A}_n$ leave the center of mass invariant, so if $y(n)$ exists, it must be $1/n$. Here is a formal argument. The attractor point $y(n)$ satisfies the recurrence

$$y(n) = \rho y(n-1) + (1-\rho y(n-1)-\rho)y(n),$$

where, for consistency, $y(1) = 1$. This implies that

$$\frac{1}{y(n)} = 1 + \frac{1}{y(n-1)};$$

therefore $y(n) = 1/n$, as claimed. The total s-energy $E(n,s)$ satisfies the relation $E(1,s) = 0$, and, for $n > 1$,

$$\begin{aligned} E(n,s) &= \rho^s E(n-1,s) + (1 - \rho y(n-1) - \rho)^s E(n,s) + 1 \\ &\geq \frac{\rho^s E(n-1,s) + 1}{1 - (1-2\rho)^s} \geq \frac{\rho^{(n-2)s}}{(1-(1-2\rho)^s)^{n-1}}\,. \end{aligned}$$

Since ρ is small enough, $(1-2\rho)^s \geq 1 - 3\rho s$ and $E(n,s) \geq s^{1-n}\rho^{-\Omega(n)}$ for any n large enough, $s \leq s_0$, and fixed $s_0 < 1$. We observe that algorithm $\mathcal{A}_n$ cannot start the second recursive call before the first one is finished, which literally takes forever. This technicality is easily handled, however, and we skip the discussion. This completes the proof of the lower bound of Theorem 1.1 for $s < 1$. ∎

The case $s = 1$. Suppose that each G_t consists of two nodes joined by an edge. The length of the edge can be made to shrink by a factor of $1 - 2\rho$. We show that having n agents allows us to mimic the behavior of a two-agent system with ρ replaced by (roughly) ρ^n; in other words, contraction can be made to slow down exponentially in n. Without loss of generality, we assume that n is an even integer $2m \geq 4$. Our construction is symmetric by reflection along the X-axis about the origin, so we label the agents $-m, \ldots, -1, 1, \ldots, m$ from left to right and restrict our discussion to the m agents with positive coordinates. (Equivalently, we could fix one agent.) The evolution of the system consists of phases denoted by $\theta = 0, 1, 2$, etc. At the beginning of phase θ, agent i lies at $x_1(\theta) = (1-\rho^m)^\theta$ for $i = 1$ and at[6]

$$x_i(\theta) = x_{i-1}(\theta) + \rho^{i-1}(1-\rho^m)^\theta$$

for $2 \leq i \leq m$. As usual, we assume that $\rho > 0$ is small enough. The system includes a mirror image of this configuration about the origin at all times. Note that all the agents are comfortably confined to the interval $[-2, 2]$, so the diameter D is at most 4.

We now describe the motion at phase θ in chronological order, beginning with agent m. During phase θ, the first graph G_t ($t = \theta m$) consists of two edges: one joins m and $m-1$; the other is its mirror image across $x = 0$. The last graph in phase θ, G_{t+m-1}, follows a different pattern: it joins the two agents indexed 1 and -1. Except for m, all of these agents (to the right of the origin) are moved twice during phase θ: first to the right, then to the left. Specifically, agent $1 \leq i < m$ moves right at time $t + m - i - 1$ and left at time $t + m - i$. We use barred symbols to denote the intermediate states, i.e., the location after the rightward motions. At phase θ,

$$G_t : \begin{cases} x_m(\theta+1) = \alpha_m x_{m-1}(\theta) + (1-\alpha_m)x_m(\theta) = (1-\rho^m)x_m(\theta), \\ \bar{x}_{m-1}(\theta) = \frac{1}{2}x_{m-1}(\theta) + \frac{1}{2}x_m(\theta) = x_{m-1}(\theta) + \frac{1}{2}\rho^{m-1}(1-\rho^m)^\theta\,. \end{cases}$$

We easily verify the identities above for $\alpha_m = (\rho - \rho^{m+1})/(1-\rho)$. For $i = m-1, m-2, \ldots, 2$, with G_{t+m-i} joining agents $i-1$ and i, the two moves are specified by

$$G_{t+m-i} : \begin{cases} x_i(\theta+1) = \alpha_i x_{i-1}(\theta) + (1-\alpha_i)\bar{x}_i(\theta) = (1-\rho^m)x_i(\theta)\,, \\ \bar{x}_{i-1}(\theta) = (1-\beta_i)x_{i-1}(\theta) + \beta_i \bar{x}_i(\theta) = x_{i-1}(\theta) + \frac{1}{2}\rho^{i-1}(1-\rho^m)^\theta\,, \end{cases}$$

[6]We deviate slightly from our usual notation by letting the argument of $x_i(\theta)$ refer to the phase of the construction and not the time t.

where $\beta_i = 1/(2+\rho)$ and

$$\alpha_i = \frac{\rho}{2+\rho} + \frac{2(1-\rho^i)\rho^{m-i+1}}{(1-\rho)(2+\rho)}\,.$$

Finally, at time $t+m-1$, choosing $\alpha_1 = (\rho + 2\rho^m)/(4+2\rho)$ allows us to write

$$G_{t+m-1}:\ x_1(\theta+1) = -\alpha_1\bar{x}_1(\theta) + (1-\alpha_1)\bar{x}_1(\theta) = (1-\rho^m)x_1(\theta)\,.$$

All the coefficients α_i are $\Theta(\rho)$, so we can rescale ρ by a constant factor to make the dynamics conform to a standard one-dimensional bidirectional agreement system with parameter ρ (and likewise ensure unit diameter). Obviously the system converges to consensus. In each phase θ, the union of the intervals formed by the edges of all of that phase's graphs G_t covers $[-x_m(\theta), x_m(\theta)]$; therefore, the total 1-energy is at least

$$2\sum_{\theta=0}^{\infty} x_m(\theta) = \frac{2(1-\rho^m)}{1-\rho}\sum_{\theta=0}^{\infty}(1-\rho^m)^\theta > \rho^{-m}.$$

This proves the lower bound of Theorem 1.1 for $s=1$. For any positive $\varepsilon < 1/2$, the length of the edge in G_{t+m-1}, which is $2x_1(\theta)$, does not fall below ε until θ is on the order of $\rho^{-m}\log\frac{1}{\varepsilon}$, which establishes the lower bound of Theorem 1.4. We note that the first agent oscillates around its initial position by roughly $\rho/2$ until θ reaches $\rho^{-\Omega(n)}$, so the kinetic 1-energy is, like the total 1-energy, exponential in n. □

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