Bulow & Klemperer, 94 "Rational Frenzies & Crashes"

asset markets are volatile!

common wisdom → irrational behavior
market imperfections

this paper offers a model (simple situation) in which rational behavior leads to "frenzies" and "crashes."

Key ideas use equilibrium strategy in auctions + the revenue equivalence theorem as the basis for buyer actions — in a dynamic setting.

WTP = "Willingness to Pay" changes from moment to moment & can change suddenly & drastically based on newly revealed information.

The Model

K identical units for sale

K+L risk-neutral potential buyers, each wants a single unit (important)

IPVs, drawn from a distribution $F(v)$ on $[0, \bar{V}]$

Buyer derives surplus $v - p$ from a purchase at price p

Dynamics

(1) Seller begins offering units at max price $\bar{V}$ and lowers it until a purchase occurs, at price p.

(2) (NEW SALE) When a purchase occurs, every buyer gets an invitation to purchase 1 unit at price p. Either

(a) (FRENZY) all goods are sold at p $\rightarrow$ game ends

(b) (FRENZY) not all goods are sold at p, no one is left willing to buy at that price.
$\rightarrow$ then go to (1) and continue lowering price until another NEW SALE takes place.

(c) (EXCESS DEMAND) More buyers want to buy at price p than there are units remaining.
Then if there are $k+l$ bidders offering to buy ~~all~~ the remaining k units, we go to (1) and restart the game with these $k+l$ bidders competing for the remaining k units. All previous sales remain valid.

We restrict ourselves to symmetric equilibria in which bidders never bid more than their value.

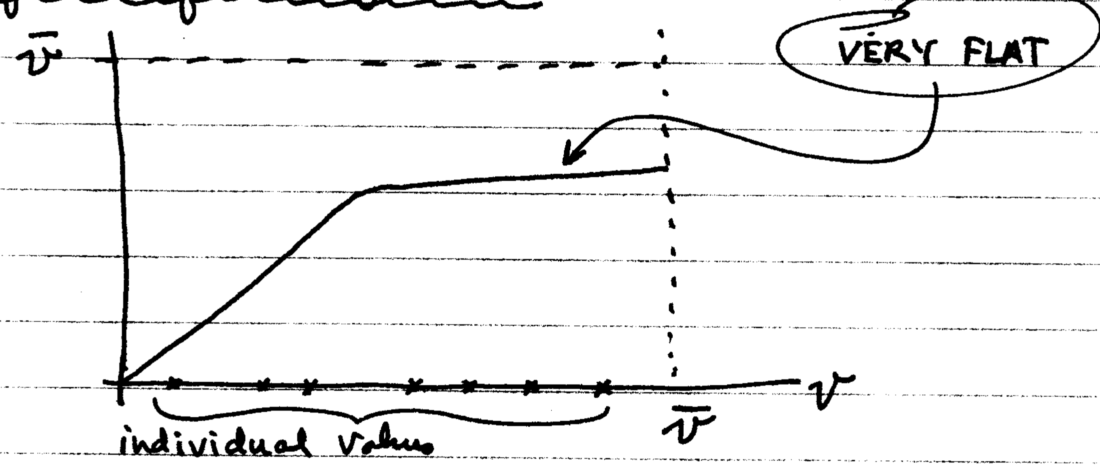
Solution to game   Apply auction theory!

at any point, k = # units remaining
$k+l$ = total # of bidders remaining
($l = L$ unless a case (c) restart has occurred)
$\bar{v}$ = highest possible value of the remaining bidders, conditional on the bidders following equilibrium strategies.
p = current asking price.

Key fctn: $w(v) = E$[price a bidder would pay in a first-rejected-price auction | winning an object]

$= E$[$(k+1)$st highest out of $(k+l)$ values | $(k+1)$st highest $\leq v$]

Important feature of $w(v)$:



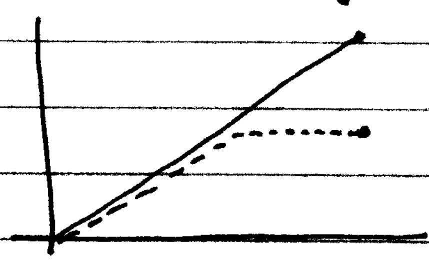
Why? → in some range of v, buyers are fairly certain of being above $(k+1)$st highest — doesn't change much in this range.

(Slight generalization of Riley & Samuelson et al.)
<u>Revenue Equivalence Theorem</u> Any equilibrium in class $\mathcal{C}$ has expected payment conditional on winning an item = $\omega(v)$.

Leads to <u>Equilibrium Strategy</u>: offer to buy if and only if $\omega(v) \geq p$.

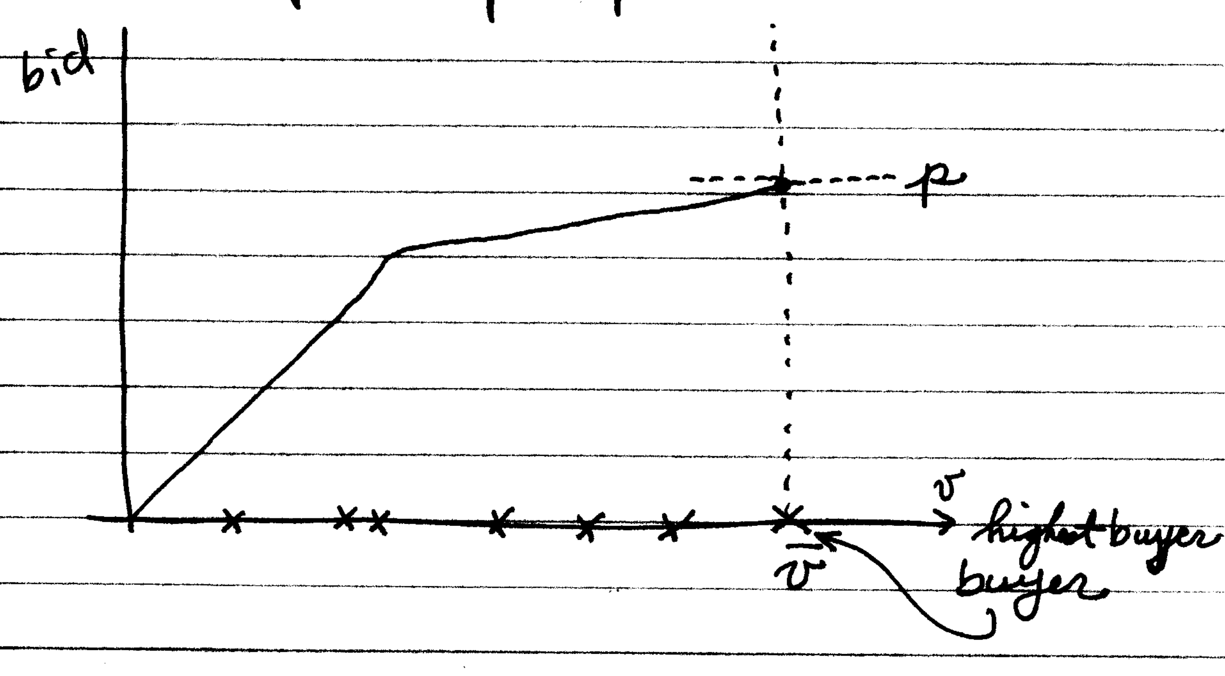
Hence to term $\omega(v) =$ WTP = willingness to Pay

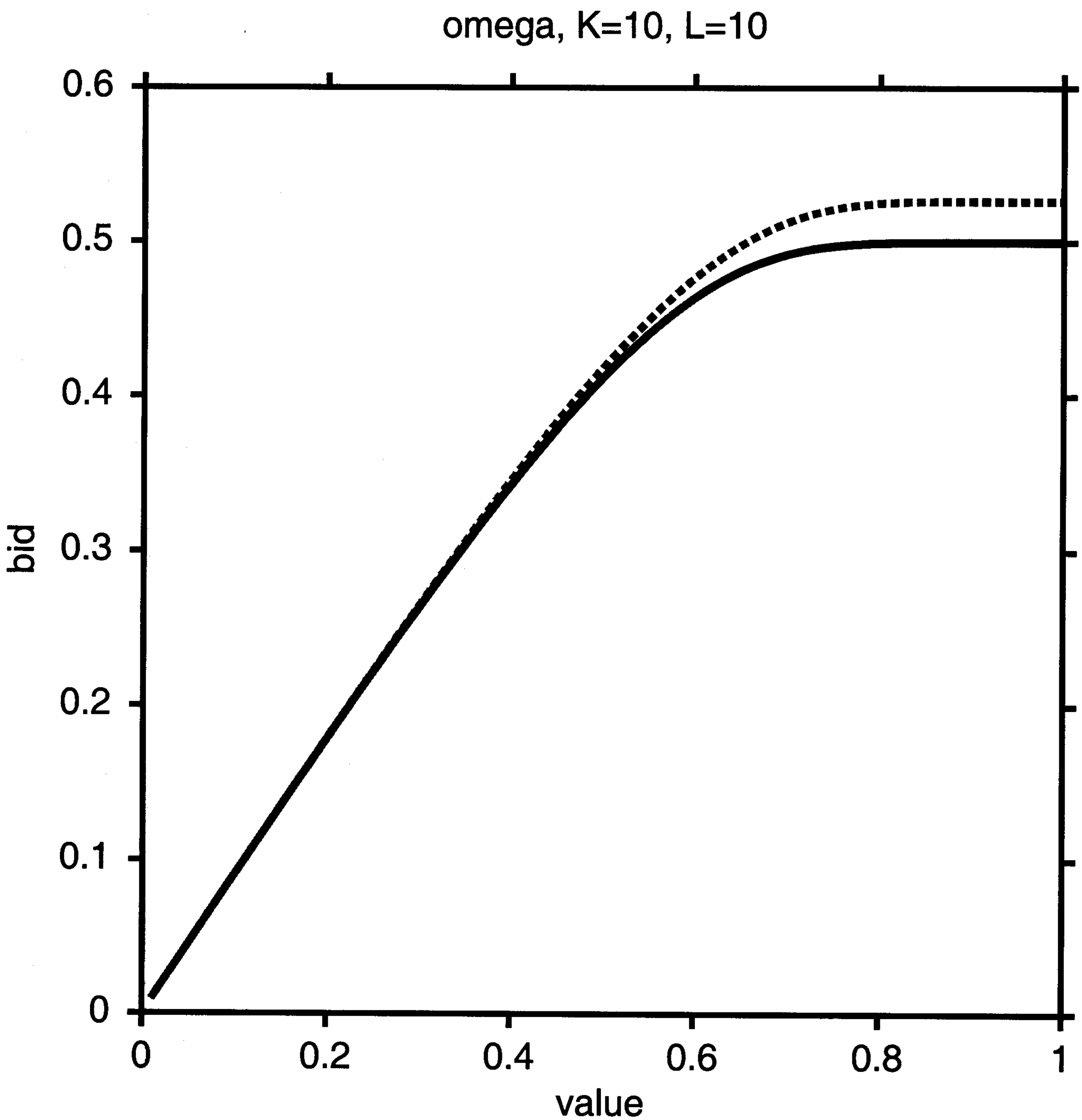
Contrast with willingness to accept <u>take-it-or-leave-it final offers</u>! $\longrightarrow$ just $\hat{\omega}(v) = v \geq p$.

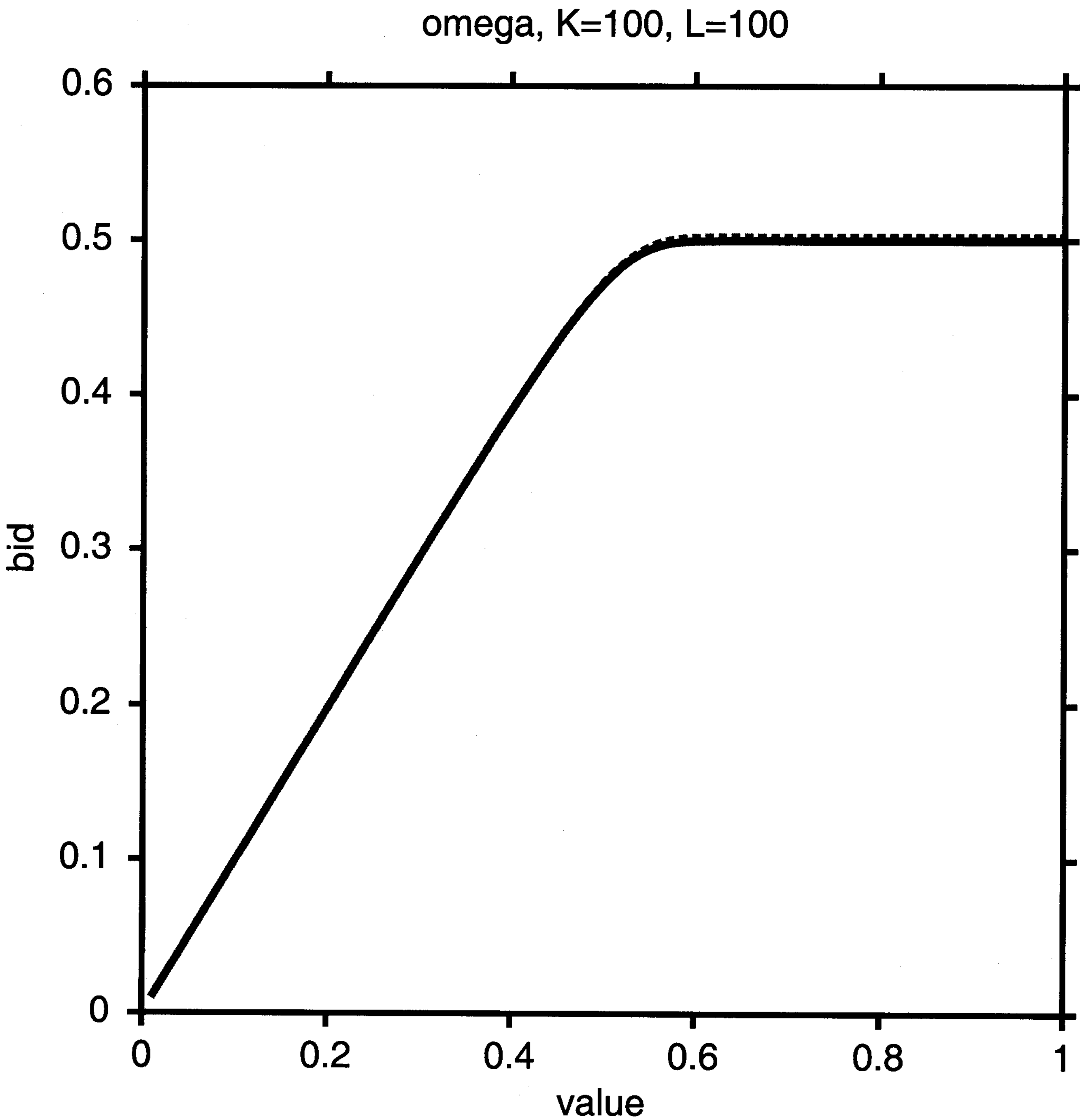


<u>Dynamics of Frenzy</u>: <u>why do others jump in?</u>

New Sale takes place at price p

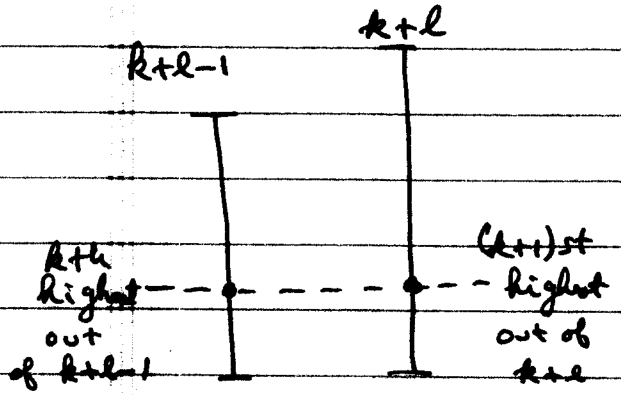



omega, K=10, L=10
0.6
0.5
0.4
0.3
0.2
0.1
0
bid
0
0.2
0.4
0.6
0.8
1
value


omega, K=100, L=100
bid
value
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1

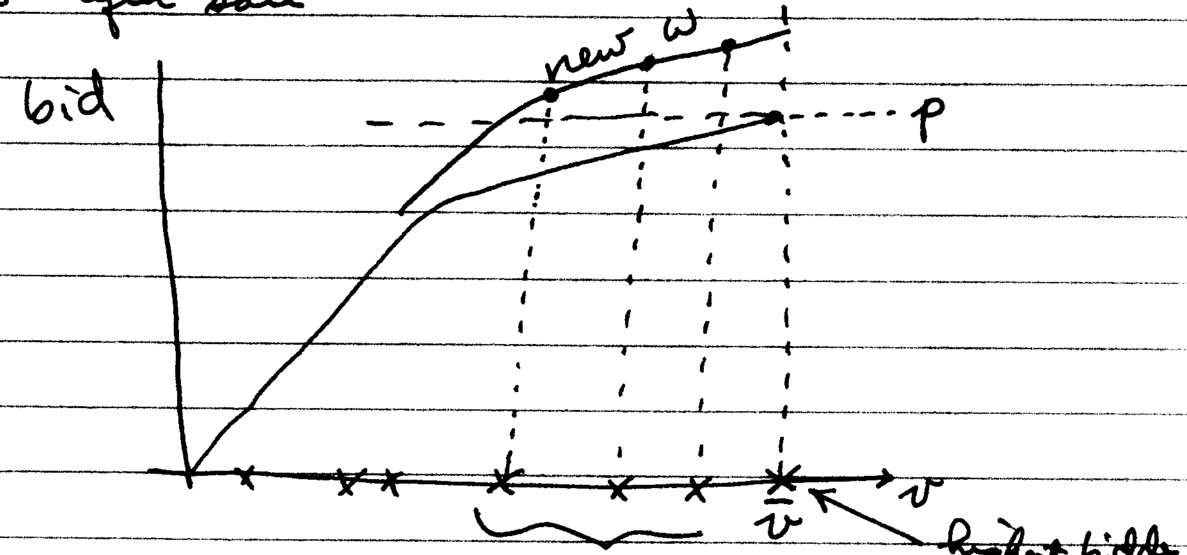
after a sale, $k = k-1$.

this ~~raises~~ $w(v)\ \forall v$:



$E[\ ] > E[\ ]$

So after sale



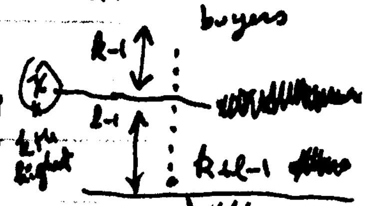
~~these bidders~~
jump in $\Rightarrow$ FRENZY

If w is very flat, this can result in a large initial frenzy, set off by one purchase.

Corrected 4/18/01
— phone rjm & Bulow

Derivation of $\omega(v)$

k items remain
$k+l$ = total # buyers



prob. k~~th~~ highest is between x & $x + dx$

$= [f(x)\cdot dx]$ prob. [exactly $k-1$ are $> x$]

$= [f(x)\cdot dx]$ [# choice for k~~th~~ highest] · prob [a given set of $(k-1)$ are $> x$ & a given set of $(l-1) < x$]

$$= f(x)\cdot dx \cdot (k+l)(k+l-1)\cdots(l) \cdot [1-F(x)]^{k-1}[F(x)]^{l-1}$$

-1

a check: $k = 1$ $\quad f(x)\cdot dx \cdot n \cdot [F(x)]^{n-1}$ ✓

$k = 2$ $\quad f(x)\cdot dx \cdot n(n-1)\cdot [1-F(x)]^{1}[F(x)]^{n-2}$ ✓

$n = k+l-1$

this gives $g(x)$ & $G(x)$ is prob. distr. of k~~th~~st order statistic.

From this, the conditional expectation ω is known!

$$\omega(v) = E[\, k\text{~~th~~ highest out of } k+l-1 \mid k\text{~~th~~ highest} \le v \,]$$

$$= \frac{\int_0^v x\cdot g(x)\,dx}{\text{prob}[\,k\text{st highest} \le v\,]}$$

$$= \int_0^v x\cdot g(x)\,dx \Big/ \int_0^v g(x)\,dx$$

$$= \frac{\int_0^v x\cdot f(x)\cdot[1-F(x)]^{k-1}[F(x)]^{l-1}\,dx}{\int_0^v f(x)\,[1-F(x)]^{k-1}[F(x)]^{l-1}\,dx}$$

k^{th} highest out of other $k+l-1$ $\le v \Rightarrow$ v is among k highest

this k^{th} highest becomes $(k+1)$st highest

B&K has $[1-F(x)]^{k-1} \ldots$? I'll use that for now

In uniform case $F(x) = x$, $f(x) = 1$.

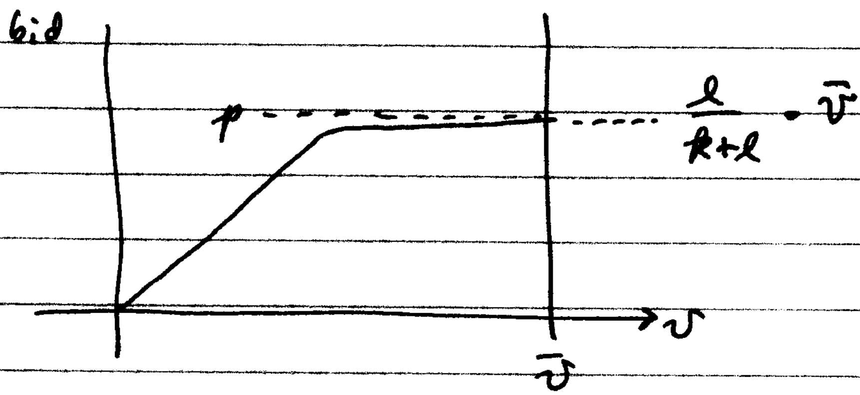
these are integrals $\int_0^{v} [1-x]^{k-1} x^{l-1}\, dx$

are called incomplete β fctns. — go back to Laplace, tabled by Pearson.

when $v = \bar{v} = 1$,

$$\omega(\bar{v}) = \omega(1) = \frac{l}{k+l}$$

So this looks like



[ω is really a fctn. $\omega(l, k, \bar{v}, v)$.
Easy to verify that
$$\omega(l, k, \bar{v}, v) = \bar{v} \cdot \omega(l, k, 1, v/\bar{v})$$]

TABLES OF THE INCOMPLETE BETA-FUNCTION

Originally prepared under the direction of and edited by

KARL PEARSON

SECOND EDITION

with a new Introduction by

E.S.PEARSON *and* N.L.JOHNSON

CAMBRIDGE

Published for the Biometrika Trustees

AT THE UNIVERSITY PRESS

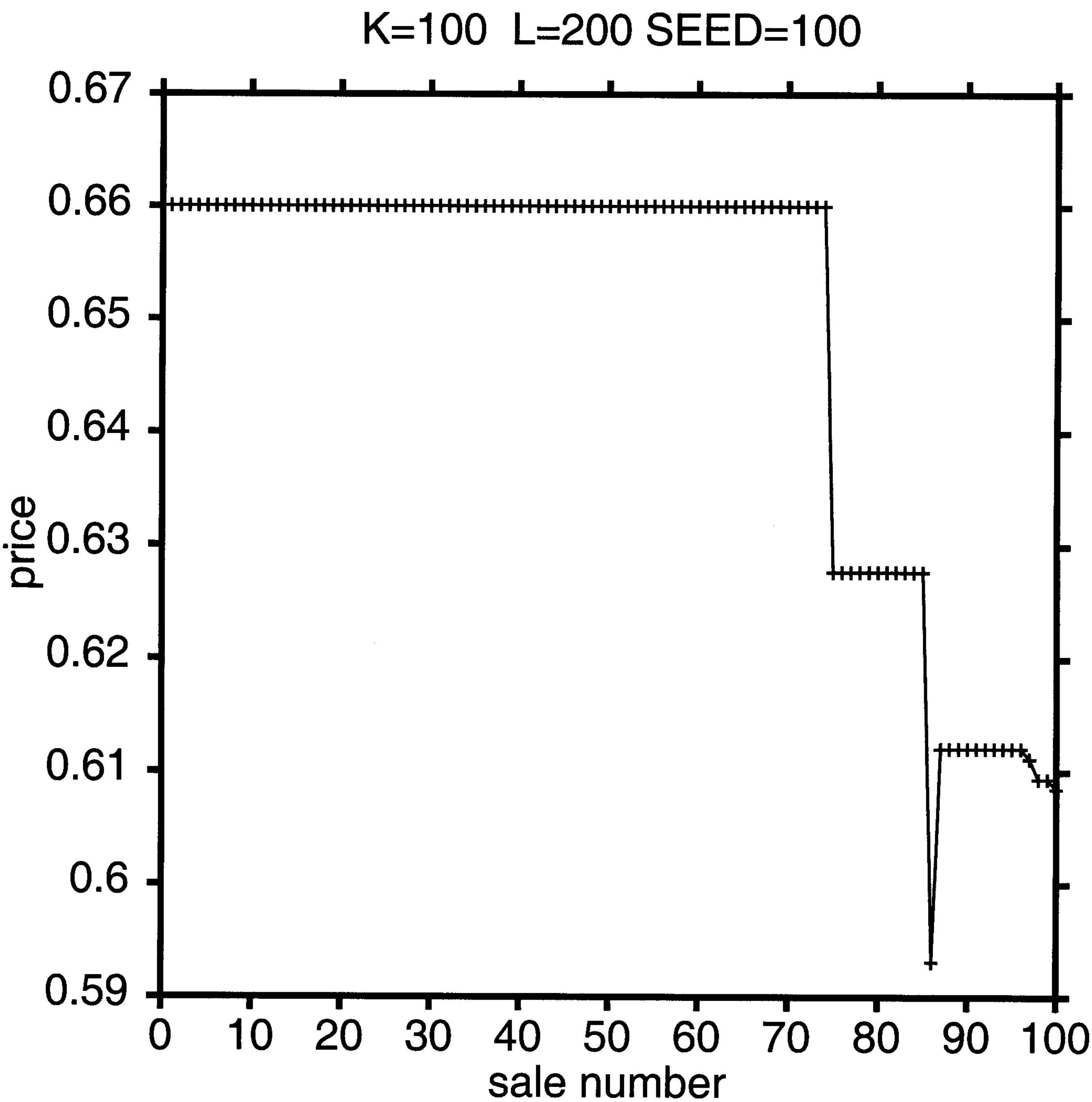
1968

sec per evaluation

a	b	x	Simpson's Rule
5	5	0.1	0.5e-2
10	10	0.2	0.9e-2
100	100	0.2	0.15
200	200	0.3	0.17
300	300	0.1	0.62e-1
300	300	0.2	0.44
300	300	0.3	0.4
400	100	0.3	0.9
400	100	0.1	---

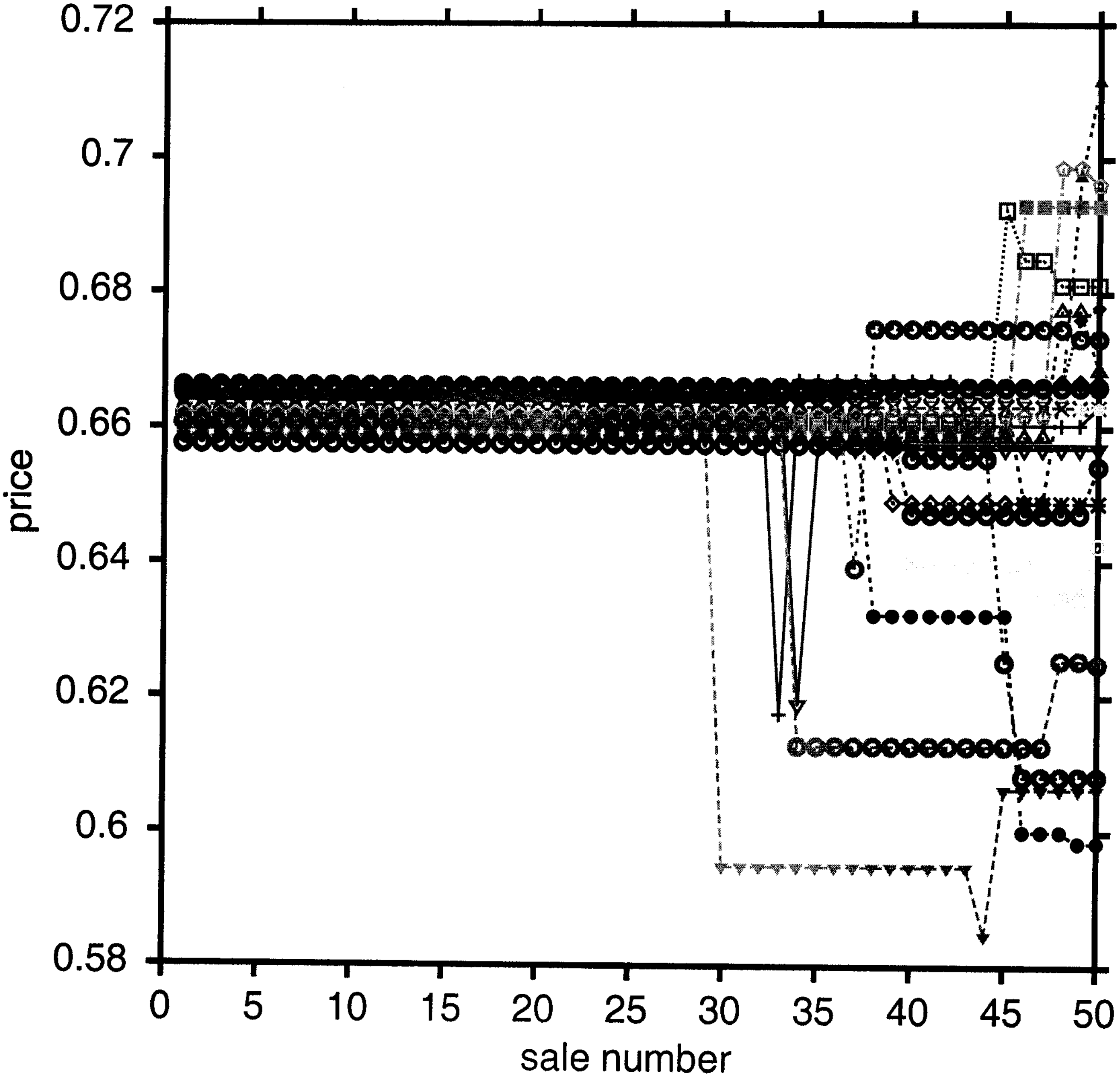
sec per evaluation

a	b	x	cont. frac.
5	5	0.1	0.7e-5
10	10	0.2	0.8e-5
100	100	0.2	0.8e-5
200	200	0.3	0.9e-5
300	300	0.1	1.1e-5
300	300	0.2	0.7e-5
300	300	0.3	0.9e-5
400	100	0.3	0.7e-5
400	100	0.1	0.6e-5


K=100 L=200 SEED=100
price
0.67
0.66
0.65
0.64
0.63
0.62
0.61
0.6
0.59
0
10
20
30
40
50
60
70
80
90
100
sale number

```
==================================================
K= 100 L= 200 random seed= 100
transaction code c: s = single sale, r = frenzy, round r, e excess demand
c   i  k  l     vi      pi      p
s   0 100 200 0.9901 0.6600 0.6600

vtilde after single sale= 0.7094
-   1 99 200 0.9871 0.6623 0.6600
...
-  73 99 200 0.7179 0.6611 0.6600
X  74 99 200 0.7093 0.6600 0.6600
         size = 73 --> fillable positive demand

FRENZY, round 1
f   1 99 200 0.9871 0.6623 0.6600
...
f  72 28 200 0.7235 0.6616 0.6600
f  73 27 200 0.7179 0.6611 0.6600
END FRENZY, round 1
new vsup after round 1 0.7094
after round 1, p is above right asymptote

X  74 26 200 0.7093 0.6278 0.6600
         size = 0

s  74 26 200 0.7093 0.6277 0.6277

vtilde after single sale= 0.6486
-  75 25 200 0.7073 0.6305 0.6277
...
-  84 25 200 0.6538 0.6291 0.6277
X  85 25 200 0.6376 0.6226 0.6277
         size = 10 --> fillable positive demand

FRENZY, round 1
f  75 25 200 0.7073 0.6305 0.6277
...
f  84 16 200 0.6538 0.6291 0.6277
END FRENZY, round 1
new vsup after round 1 0.6486
after round 1, p is above right asymptote

X  85 15 200 0.6376 0.6033 0.6277
         size = 0

s  85 15 200 0.6376 0.5931 0.5931

vtilde after single sale= 0.6067
-  86 14 200 0.6372 0.5959 0.5931
...
- 102 14 200 0.6068 0.5931 0.5931
X 103 14 200 0.6038 0.5917 0.5931
         size = 17 --> excess demand, new vund = 6.066653e-01

s  86 14   3 0.6372 0.6120 0.6120

price increase after excess demand, seed= 100, 87/100 items sold
vtilde after single sale= 0.6180
-  87 13   3 0.6346 0.6124 0.6120
...
-  95 13   3 0.6221 0.6123 0.6120

X  96 13   3 0.6171 0.6119 0.6120
         size = 9 --> fillable positive demand

FRENZY, round 1
f  87 13   3 0.6346 0.6124 0.6120
...
f  94  6   3 0.6225 0.6124 0.6120
f  95  5   3 0.6221 0.6123 0.6120
END FRENZY, round 1
new vsup after round 1 0.6180
after round 1, p is above right asymptote

X  96  4   3 0.6171 0.6115 0.6120
         size = 0

s  96  4   3 0.6171 0.6111 0.6111

vtilde after single sale= 0.6136
X  97  3   3 0.6121 0.6103 0.6111
         size = 0

s  97  3   3 0.6121 0.6094 0.6094

vtilde after single sale= 0.6106
-  98  2   3 0.6111 0.6097 0.6094
X  99  2   3 0.6091 0.6084 0.6094
         size = 1 --> fillable positive demand

FRENZY, round 1
f  98  2   3 0.6111 0.6097 0.6094
END FRENZY, round 1
new vsup after round 1 0.6106
new vtilde after round 1 0.6103

X  99  1   3 0.6091 0.6085 0.6094
         size = 0

s  99  1   3 0.6091 0.6085 0.6085

K items sold, sale over
average price per item=   0.648998
```

K= 50, L=100


0.72
0.7
0.68
0.66
0.64
0.62
0.6
0.58
price
0
5
10
15
20
25
30
35
40
45
50
sale number

10000 runs, price per item

tests of Revenue Equiv.

k	l	avg	predicted
1	1	0.333347	0.333333
1	2	0.500026	0.500000
2	2	0.400157	0.400000
1	3	0.600616	0.600000
2	3	0.499878	0.500000
1	4	0.666849	0.666666
20	10	0.322530	0.322580
30	10	0.243460	0.243902
40	5	0.108449	0.108695
5	40	0.869741	0.869565
20	30	0.588372	0.588235
1	100	0.980528	0.980392
100	100	0.497485	0.497512
200	100	0.332154	0.332225

RevCheck checks total expected revenue in uniform case against what is known from revenue equivalence theorem: that E[revenue] in equilibrium = same as SP auction = E[(K+L)st highest of K+L bidders]. When K=1 object and L=1, we have K+L=2 bidders and 1 object, and (L+1)st highest out of 3 has exp. value 1/3. In general, L = no. of bidders - no. of objects, and the exp. revenue is L/(K+L+1).

Closed form for E[price]

5/12/01

1

$$E[\text{price}] = \frac{(k+l)!}{k(l-1)!}\int_0^1\left[\int_0^V (1-x)^{k-1} x^{l}\,dx\right] dV$$

$$= \frac{(k+l)!}{k(l-1)!}\left\{\left[\int_0^V (1-x)^{k-1}x^{l}\,dx\right]\cdot V\Big|_0^1 - \int_0^1 V^{l+1}(1-V)^{k-1}\,dV\right\}$$

$$= \frac{(k+l)!}{k(l-1)!}\left[\int_0^1 (1-x)^{k-1}x^{l}\,dx - \int_0^1 V^{l+1}(1-V)^{k-1}\,dV\right]$$

$$\int_0^1 t^{z-1}(1-t)^{w-1}\,dt = B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

$$= \frac{(z-1)!\,(w-1)!}{(z+w-1)!}$$

$$E[\text{price}] = \frac{(k+l)!}{k(l-1)!}\left[\frac{(k-1)!\,l!}{(k+l+1-1)!} - \frac{(k-1)!\,(l+1)!}{(k+l+1)!}\right]$$

$$= \frac{(k+l)!}{k(l-1)!}\left[\frac{(k-1)!\,l!}{(k+l)!} - \frac{(k-1)!\,(l+1)!}{(k+l+1)!}\right]$$

$$= \frac{\cancel{(k+l)!}\,(k-1)!\,l!}{k\,(l-1)!\,\cancel{(k+l)!}}\left[1 - \frac{l+1}{k+l+1}\right]$$

$$= \frac{l}{k}\left[1 - \frac{l+1}{k+l+1}\right] = \frac{l}{k}\left[\frac{\cancel{k}+\cancel{l}+\cancel{1}-\cancel{l}-\cancel{1}}{k+l+1}\right] = \boxed{\frac{l}{l+k+1}}$$

<u>check</u>

$k=1,\ l=1$	$\frac{1}{3}$ ✓	
$k=1,\ l=2$	$\frac{2}{4}$ ✓	OK
$k=2,\ l=2$	$\frac{2}{5}$ ✓	
$k=1$	$\frac{l}{l+2}$ ✓	
$k=2,\ l=3$	$\frac{3}{6} = 1/2$ ✓	