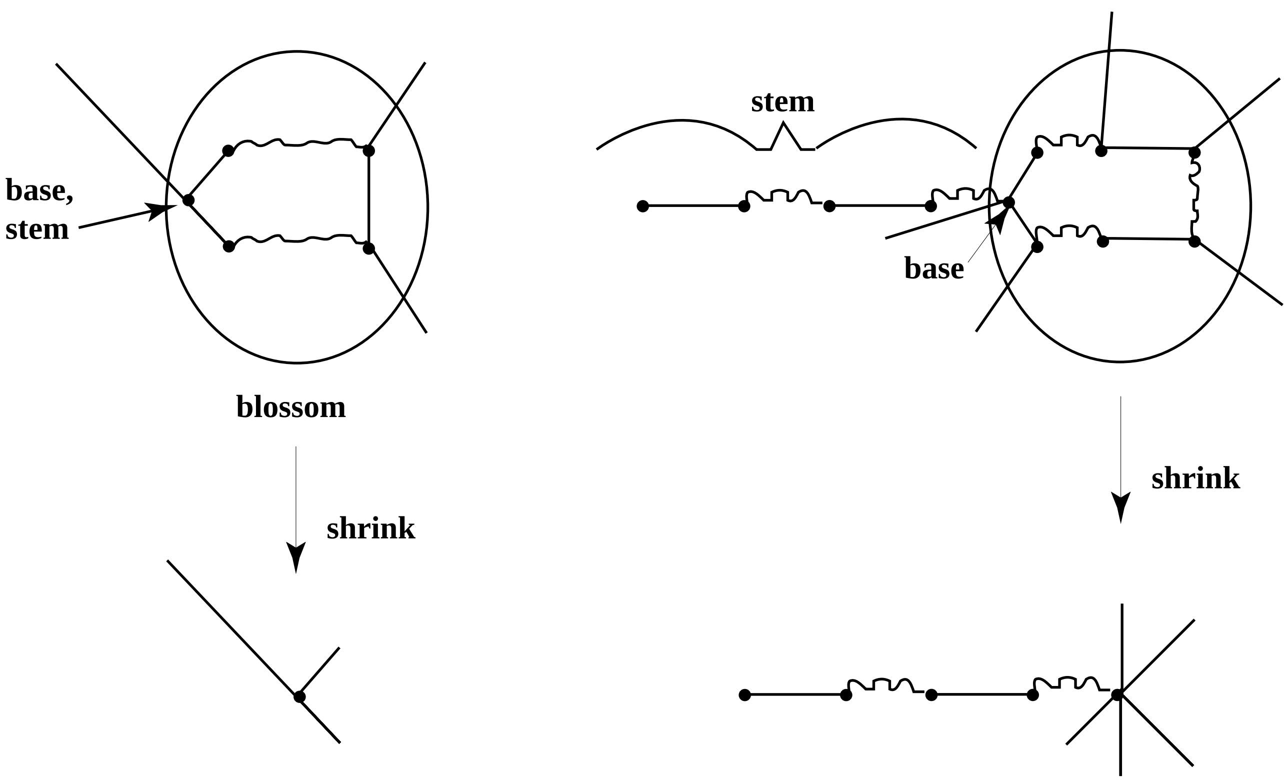
COS 423: Theory of Algorithms
Spring, 2000 — Tarjan
Notes #4

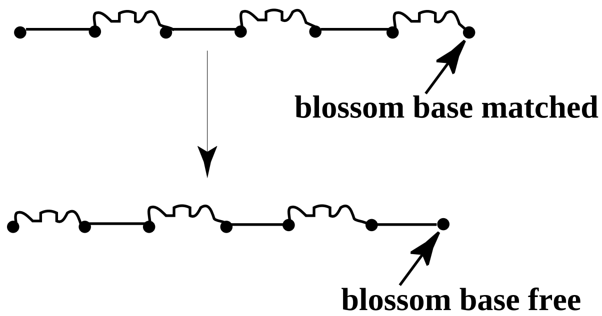
Sketchy Notes on Edmonds' Incredible Shrinking Blossom Algorithm for General Matching

Consider an alternating even-length path P from a free vertex v to a vertex w plus an odd-length alternating cycle from w to itself. Cycle B is a **blossom**; path P is a **stem**; vertex w is the **base** of the blossom. **Shrinking** the blossom consists of contracting all vertices of B into a single vertex.



Theorem: Let G' be formed from G by shrinking a blossom B. Then G' contains an augmenting path iff G does.

Proof: If the base w of the blossom B is not free, change the matching in G by switching the edges along the stem to make w free (see figure below). Then B is a blossom with a free vertex as a base. Let G_1 be G after this change and G_1' the graph resulting from shrinking B in G_1. (G and G_1, and also G' and G'_1, differ only in which edges are matched and which are unmatched). The switching does not change the matching size; thus G has an augmenting path iff G_1 does, and G' has iff G'_1 does. Thus we need consider only the case in which the base of the blossom is free.



Suppose G' has an augmenting path. Either this path is an augmenting path in G, or it ends at the blossom, and it can be extended to an augmenting path in G by following the blossom in the direction that results in alternation until reaching the base.

Suppose G has an augmenting path. Either it is an augmenting path in G' or it hits the blossom, in which case the part from one end until the blossom is first hit is an augmenting path in G'.

Edmonds' algorithm to find an augmenting path or shrink a blossom

Start with all vertices unlabeled and all edges unexamined. Repeat steps until finding a blossom, finding an augmenting path, or running out of unlabeled free vertices and unexamined edges incident to even vertices.

Either: Choose an unlabeled free vertex v. Label it $[v, \text{even}]$.

Or: Choose an unexamined edge $\{v, w\}$ with v labeled $[r, \text{even}]$. Mark the edge examined.

If w is unlabeled and free, stop: augmenting path found.

If w is unlabeled and matched to x, label w $[r,\ \text{odd}]$ and x $[r,\ \text{even}]$.

If w is labeled $[s,\ \text{even}]$ with $r \neq s$ stop: augmenting path found.

If w is labeled $[r,\ \text{even}]$ stop: blossom found.

(If w is labeled $[s,\ \text{odd}]$, do nothing.)

On finding a blossom, shrink it and restart.

On finding an augmenting path, expand all blossoms in reverse order of shrinking, adding edges to the augmenting path to keep it an augmenting path after each blossom expansion. Having found an augmenting path in the original graph, switch matched and unmatched edges along it to increase the size of the matching by one. Restart.

Proof of correctness: Blossom-shrinking preserves the existence or non-existence of an augmenting path. Suppose there is an augmenting path. Then the algorithm will not stop before both ends of it are labeled even. Since each matched edge $\{v, w\}$ has one end labeled odd and one even, or both unlabeled, there must be an unmatched edge $\{v, w\}$ along the augmenting path with v labeled even and w labeled even. But this edge will be examined by the algorithm before the algorithm stops; and, when it is, either a blossom or an augmenting path will be found.